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Perfect 3-Colorings of some Durer Graphs

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Abstract

The notion of a perfect coloring, introduced by Delsarte, generalizes the concept of completely regular code. A perfect $z-$colorings of a graph is a partition of its vertex set. It splits vertices into z parts $P_1, \ldots, P_z$ such that for all $i, j \in \{1, \ldots, z\}$, each vertex of P_i is adjacent to p_{ij}, vertices of P_j. The matrix $P = (p_{ij})_{i,j \in \{1,\ldots,z\}}$, is called parameter matrix. In this article, we classify all the realizable parameter matrices of perfect 3-colorings of some durer graphs.

Keywords: Parameter matrices, Perfect coloring, Equitable partition, Durer graph.
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1 Introduction

The concept of a perfect z-coloring plays a significant role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another phrase for this concept in the writing as "equitable partition" ([9]). In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Since then, some effort has been made to count the parameter matrices of some Johnson graphs, including $J(4, 2), J(5, 2), J(6, 2), J(6, 3), J(7, 3), J(8, 3), J(8, 4)$, and $J(v, 3)$ (v odd) ([3, 4, 8]).

Fon-Der-Flaass count the parameter matrices (perfect 2-colorings) of n-dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix ([5, 6, 7]). In this article, we classify the parameter matrices of all perefect 2-colorings of some durer graphs.

The durer graph is the skeleton of dürer's solid, which is the generalized peterson graph $Gp(6, 2)$. Some durer graphs given as follow:

[1]speaker

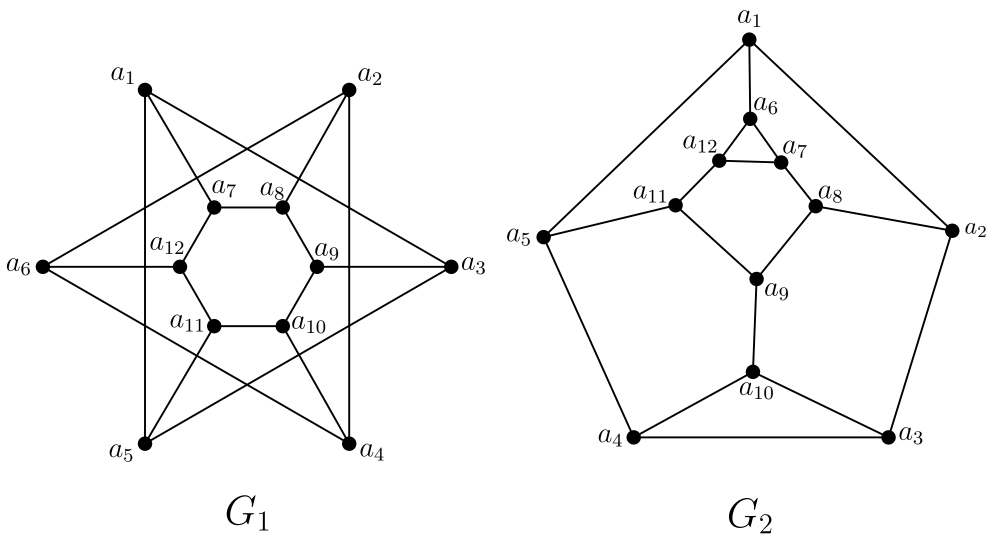

Figure 1: Some durer graphs

Definition 1.1. For a graph G and an integer z, a mapping $T : V(G) \longrightarrow \{1, \ldots, z\}$ is called a perfect z-coloring with the matrix $P = (p_{ij})_{i,j \in \{1,\ldots,z\}}$, if it is surjective, and for all i, j, for every vertex of color i, the number of its neighbours of color j is equal to p_{ij}. The matrix P is called the parameter matrix of a perfect coloring. In the case $z = 3$, we call the first color white that show by W, the second color black that show by B and the third color red that show by R. In this article, we generally show a parameter matrix by

$$P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Remark 1.2. In this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e. We identify the perfect 3-colorings with the matrix

$$\begin{bmatrix} d & c & b \\ g & i & h \\ d & e & f \end{bmatrix}, \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix}, \begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix}, \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}, \begin{bmatrix} i & g & h \\ c & a & b \\ f & d & e \end{bmatrix}.$$

obtained by switching the colors with the original coloring.

2 Preliminaries

In this section, we present some results concerning necessary conditions for the existence of perfect 3-colorings of some durer graphs with a given parameter matrix

$$P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The simplest necessary condition for the existence of perfect 3-colorings of the durer graphs

$$a + b + c = d + e + f = g + h + i = 3.$$

By using this condition and some computation, it is clear that we should consider 18 matrices. These matrices are listed below:

$$P_1 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, P_4 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}, P_5 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, P_6 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix},$$
$$P_7 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, P_8 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}, P_9 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, P_{10} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}, P_{11} = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, P_{12} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix},$$
$$P_{13} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, P_{14} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, P_{15} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, P_{16} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, P_{17} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, P_{18} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

Theorem 2.1. *[9] If T is a perfect coloring of a graph G with z colors, then any eigenvalue of T is an eigenvalue of G.*

Theorem 2.2. *[1] Suppose that T is a perfect 3- coloring with matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, in the connected graph G. Then in this case, none of the following situations will occer.*

(1) $b = c = 0$,

(2) $d = f = 0$,

(3) $g = h = 0$,

(4) $b = 0 \leftrightarrow d = 0, c = 0 \leftrightarrow g = 0, h = 0 \leftrightarrow f = 0$.

Theorem 2.3. *[2] Let T a perfect 3-coloring of a graph G with matrix $P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.*

1. *If $b, c, f \neq 0$, then*
$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, \quad |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, \quad |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

2. *If $b = 0$, then*
$$|W| = \frac{|V(G)|}{\frac{c}{g} + 1 + \frac{ch}{fg}}, \quad |B| = \frac{|V(G)|}{\frac{f}{h} + 1 + \frac{fg}{ch}}, \quad |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

3. *If $c = 0$, then*
$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{bf}{dh}}, \quad |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, \quad |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{dh}{bf}}.$$

4. *If $f = 0$, then*
$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, \quad |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{cd}{bg}}, \quad |R| = \frac{|V(G)|}{\frac{g}{c} + 1 + \frac{bg}{cd}}.$$

Theorem 2.4. *[1] If $P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a parameter matrix of a k-regular graph, then the eigenvalues of P are*
$$\lambda_{1,2} = \frac{\mathrm{tr}(A) - k}{2} \pm \sqrt{\Big(\frac{\mathrm{tr}(A) - k}{2}\Big)^2 - \frac{\det(A)}{k}}, \quad \lambda_3 = k.$$

3 Perfect 3- colorings of some durer graphs

The parameter matrices of some durer graphs are enumerated in the next teorems.

Theorem 3.1. *The graph G_1 has no perfect 3-colorings.*

Proof. A parameter matrix corresponding to perfect 3-colorings of the graph G_1 may be one of the matrices $P_1, \dots, P_{18}$. By using Theorem 2.1 and Theorem 2.4, we can see that only the matrices P_3, P_4, P_5, P_6, P_{10}, P_{12}, P_{15}, P_{16}, and P_{18} can be a parameter matrices. By using Theorem 2.3, matrices P_3, P_4, P_5, P_6, P_{12}, P_{15}, P_{18} cannot be a parameter matrices, because the number of white, black and red, are not an integer. For matrix P_{10}, each vertex with color black has zero adjacent vertex with color red. Now have the following possibilities:

(1) $T(a_1) = T(a_3) = T(a_8) = R, T(a_4) = T(a_{10}) = T(a_{11}) = T(a_{12}) = B, T(a_5) = T(a_7) = T(a_9) = W$, then $T(a_2) = R$ and $T(a_6) = B$, which is a contradiction with the second row of the matrix P_{10}.

(2) $T(a_1) = T(a_9) = T(a_{11}) = W, T(a_3) = T(a_5) = T(a_{10}) = R, T(a_6) = T(a_7) = T(a_8) = T(a_{12}) = B$, then $T(a_2) = B$ and $T(a_4) = R$, which is a contradiction with the second row of the matrix P_{10}.

Similar to matrix P_{10}, we can proof for the matrix P_{16} as follows:

For matrix P_{16}, each vertex with color black has one adjacent vertix with color white and one adjacent vertex red. Now have the following possibilities:

(3) $T(a_2) = T(a_7) = T(a_9) = T(a_{12}) = R, T(a_4) = T(a_6) = T(a_{10}) = W, T(a_8) = T(a_{11}) = B$ then $T(a_1) = T(a_3) = W$ and $T(a_5) = B$, which is a contradiction with the second row of the matrix P_{16}.

(4) $T(a_2) = T(a_{12}) = R, T(a_4) = T(a_6) = T(a_8) = B, T(a_7) = T(a_9) = T(a_{10}) = W$, then $T(a_1) = T(a_3) = T(a_{11}) = R$ and $T(a_5) = B$, which is a contradiction with the second row of the matrix P_{16}. Hence graph G_1 has no perfect 3-colorings with the matrix P_{16}.

□

Theorem 3.2. *The graph G_2 has no perfect 3-colorings.*

Proof. A parameter matrix corresponding to perfect 3-colorings of the graph G_2 may be one of the matrices $P_1, \dots, P_{18}$. By using Theorem 2.1 and Theorem 2.4, we can see that only the matrices $P_3, P_4, P_5, P_6, P_{10}, P_{12}$, P_{15}, P_{16} and P_{18} can be a parameter matrices. By using Theorem 2.3, matrices $P_3, P_4, P_5, P_6, P_{12}, P_{15}, P_{18}$ cannot be a parameter matrices, because the number of white, black and red, are not an integer. For matrix P_{10}, each vertex with color black has zero adjacent vertex with color red. Now have the following possibilities:

(1) $T(a_1) = T(a_8) = T(a_{12}) = W, T(a_2) = T(a_6) = T(a_7) = R, T(a_5) = T(a_9) = T(a_{11}) = B$, then $T(a_3) = R$ and $T(a_4) = T(a_{10}) = B$, which is a contradiction with the three row of the matrix P_{10}.

(2) $T(a_1) = T(a_2) = B, T(a_5) = T(a_8) = T(a_{10}) = T(a_{12}) = W, T(a_9) = T(a_{11}) = R$, then $T(a_3) = T(a_6) = B$ and $T(a_4) = T(a_7) = R$, which is a contradiction with the three row of the matrix P_{10}. Hence graph G_2 has no perfect 3-colorings with the matrix P_{10}.

Similar to matrix P_{10}, we can proof for the matrix P_{16} as follows:

For matrix P_{16}, each vertex with color black has one adjacent vertex with color white and one adjacent vertex black, and one adjacent vertex red. Now have the following possibilities:

(3) $T(a_1) = T(a_9) = T(a_{10}) = R, T(a_4) = T(a_7) = T(a_{12}) = W, T(a_5) = T(a_6) = T(a_{11}) = B$, then $T(a_2) = B$ and $T(a_3) = T(a_8) = W$, which is a contradiction with the second row of the matrix P_{16}.

(4) $T(a_1) = T(a_6) = T(a_8) = T(a_9) = B$, $T(a_5) = T(a_7) = T(a_{11}) = R, T(a_{12}) = W$, then $T(a_3) = R$ and $T(a_2) = T(a_4) = T(a_{10}) = W$, which is a contradiction with the three row of matrix P_{16}. Hence graph G_2 has no perfect 3-colorings with the matrix P_{16}.

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