Perfect 2-Colorings of some Durer Graphs

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Abstract

The notion of a perfect coloring, introduced by Delsarte, generalizes the concept of completely regular code. A perfect $z-$colorings of a graph is a partition of its vertex set. It splits vertices into z parts $P_1, \ldots, P_z$ such that for all $i, j \in \{1, \ldots, z\}$, each vertex of P_i is adjacent to p_{ij}, vertices of P_j. The matrix $P = (p_{ij})_{i,j\in\{1,\ldots,z\}}$, is called parameter matrix. In this article, we classify all the realizable parameter matrices of perfect 2-colorings of some durer graphs.

Keywords: Parameter matrices, Perfect coloring, Equitable partition, Durer graph.
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1 Introduction

The concept of a perfect z-coloring plays a significant role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another phrase for this concept in the writing as "equitable partition" ([8]). In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Since then, some effort has been made to count the parameter matrices of some Johnson graphs, including $J(4, 2), J(5, 2), J(6, 2), J(6, 3), J(7, 3), J(8, 3), J(8, 4)$, and $J(v, 3)$ (v odd) ([2, 3, 7]).

Fon-Der-Flaass count the parameter matrices (perfect 2-colorings) of n-dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix ([4, 5, 6]). In this article, we classify the parameter matrices of all perefect 2-colorings of some durer graphs.

The durer graph is the skeleton of dürer's solid, which is the generalized peterson graph $Gp(6, 2)$. Some durer graphs given as follow:

[1]speaker

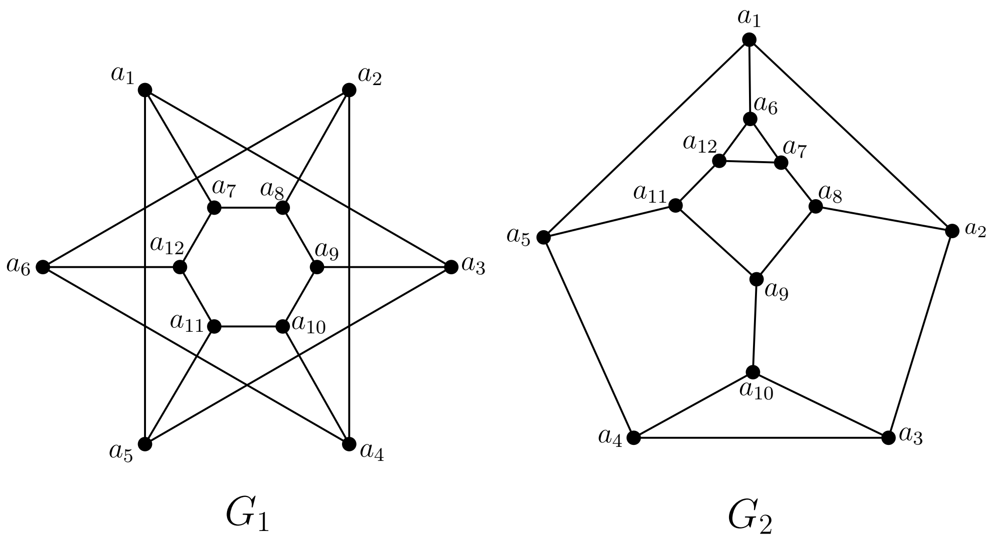

Figure 1: Some durer graphs

Definition 1.1. For a graph G and an integer z, a mapping $T : V(G) \longrightarrow \{1, \ldots, z\}$ is called a perfect z-coloring with the matrix $P = (p_{ij})_{i,j\in\{1,\ldots,z\}}$, if it is surjective, and for all i, j, for every vertex of color i, the number of its neighbours of color j is equal to p_{ij}. The matrix P is called the parameter matrix of a perfect coloring. In the case $z = 2$, we call the first color white that show by W, the second color black that show by B. In this article, we generally show a parameter matrix by

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Remark 1.2. In this paper, we consider all perfect 2-colorings, up to renaming the colors. We identify the perfect 2-colorings with the matrix $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$, obtained by switching the colors with the original coloring.

2 Preliminaries

In this section,we first given some results concerning necessary conditions for existence of perfect 2- colorings of some durer graphs with a given parameter matrix $P = (p_{ij})_{i,j=1,2}$. The simplest condition for the existence of perfect 2-colotings of some durer graphs with the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is:

$$a + b = c + d = 3.$$

By using this condition and some computation, it is clear that we should consider 6 matrices. These matrices are listed below:

$$P_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ P_2 = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix},\ P_3 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ P_4 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix},\ P_5 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix},\ P_6 = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix},$$

Theorem 2.1. *[8] If T is a perfect coloring of a graph G with z colors, then any eigenvalue of T is an eigenvalue of G.*

Theorem 2.2. *Every perfect 2-colorings of a k-regular graph with parameter matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two eigenvalues : one is k, and the other is $a - c$ such that we obviously have $a - c \neq k$.*

So, from Theorem 2.3, we conclude that $a - c$ is an eigenvalue of a k-regular connected graph which is not equal to k.

Theorem 2.3. *[1] If W is the set of white vertices in a perfect 2-colorings of a graph G with matrix $P = (p_{ij})_{i,j=1,2}$ then*

$$|W| = |V(G)|\frac{c}{b+c}.$$

3 Perfect 2- colorings of some durer graphs

The parameter matrices of durer graphs are enumerated in the next teorems.

Theorem 3.1. *The graph G_1 has a perfect 2-colorings only with the matrices P_1 and P_2.*

Proof. A parameter matrix corresponding to perfect 2-colorings of the graph G_1 may be one of the matrices $P_1, \ldots, P_6$. Using the Theorems 2.1, 2.2, and 2.3 matrices P_1 and P_2 can be a parameter matrices. Consider the mapping T_1 and T_2 as follows:

$$T_1(a_1) = T_1(a_2) = T_1(a_3) = T_1(a_4) = T_1(a_5) = T_1(a_6) = B,$$
$$T_1(a_7) = T_1(a_8) = T_1(a_9) = T_1(a_{10}) = T_1(a_{11}) = T_1(a_{12}) = W.$$

$$T_2(a_1) = T_2(a_3) = T_2(a_4) = T_2(a_6) = T_2(a_7) = T_2(a_9) = T_2(a_{10}) = T_2(a_{12}) = W,$$
$$T_2(a_2) = T_2(a_5) = T_2(a_8) = T_2(a_{11}) = B.$$

It is clear that T_1 and T_2 are perfect 2-coloring with the matrices P_1 and P_2 respectivelly. □

Theorem 3.2. *The graph G_2 has a perfect 2-colorings only with the matrices P_1 and P_2.*

Proof. A parameter matrix corresponding to perfect 2-colorings of the graph G_2 may be one of the matrices $P_1, \ldots, P_6$. Using the Theorems 2.1, 2.2, and 2.3 matrices P_1 and P_2 can be a parameter matrices. Consider the mapping T_1 and T_2 as follows:

$$T_1(a_1) = T_1(a_2) = T_1(a_5) = T_1(a_8) = T_1(a_9) = T_1(a_{11}) = W,$$
$$T_1(a_3) = T_1(a_4) = T_1(a_6) = T_1(a_7) = T_1(a_{10}) = T_1(a_{12}) = B.$$

$$T_2(a_1) = T_2(a_4) = T_2(a_5) = T_2(a_6) = T_2(a_7) = T_2(a_8) = T_2(a_9) = T_2(a_{10}) = W,$$
$$T_2(a_2) = T_2(a_3) = T_2(a_{11}) = T_2(a_{12}) = B.$$

It is clear that T_1 and T_2 are perfect 2-colorings with the matrices P_1 and P_2 respectivelly. □

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