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On Some Properties of Jacobsthal-Narayana Difference Sequence

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Abstract

In this paper we introduce a new difference sequence by using the difference relation to the Jacobsthal-Narayana sequence. We investigate some properties of this sequence. Then we represent the Binet formula and generating functions related to this sequence.

Keywords: Jacobsthal-Narayana sequence, difference sequence, generating function.

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1 Introduction

Special number sequences like as Fibonacci sequence, Pell sequence, Jacobsthal sequence, Lucas sequence and some generalizations of these sequences have attracted many authors in the area of coding theory in computer sciences, signal processing, numerical analysis, combinatory theory and matrix theory in recent years. For instance, Bueno [1] studied (k, h)-Jacobsthal sequence of the form

$$T_n = kT_{n-1} + 2hT_{n-2}. \tag{1}$$

He found a formula for the n^{th} term and sum of the first n terms of this sequence. Bozkurt [5] computed the spectral norms of some matrices connected integer sequences such as Fibonacci, Lucas, Pell and Perrin numbers. The author in [11] introduced Jacobsthal-Narayanal sequence and Jacobsthal-Lucas-Narayana sequence and studied some properties of these sequences.
The concept of difference sequences initially was introduced by H.Kizmaz[8]. For a given sequence $x = (x_n)$, difference sequence of x_n is denoted by $\Delta(x_n)$ and is defined by

$$\Delta(x_n) = x_{n+1} - x_n. \tag{2}$$

[1]speaker

The scond difference sequence of x_n is denoted by $\Delta^2(x_n)$ and is defind by

$$\Delta^2(x_n) = \Delta(x_{n+1}) - \Delta(x_n). \tag{3}$$

Also, for the integer number j, the j^{th} difference sequence is denoted by $\Delta^j(x_n)$ and is defined by

$$\Delta^j(x_n) = \Delta^{j-1}(x_{n+1}) - \Delta^{j-1}(x_n). \tag{4}$$

Note that, for $j = 0$, difference sequence $\Delta^j(x_n)$ is defined by $\Delta^0(x_n) = x_n$.
Many authors studied this notion and introduced new difference sequences. Catarino[6], introduced Pell difference sequence and has investigated some algebraic properties of this new sequence. In this note, inspiration by the work of Catarino, we introduced new difference sequence baesd on Jacobsthal-Nayana sequence, namely Jacobsthal-Narayana difference sequence. We study some algebraic properties of this sequence. Then we obtain the Binet formula and generating function related to this sequence.
For more information about Pell sequence, (k, h)-Pell sequence, Fibonacci sequence and some generalizations of these sequences we refer to [2]- [4], [6] - [10] and [12] - [14].

Now we consider the Jacobsthal-Narayana sequence [11]. Jacobsthal- Narayana sequence $\{JN_r\}$ has the recursive relation

$$JN_r = JN_{r-1} + 2JN_{r-3}, \tag{5}$$

where $JN_0 = 0, JN_1 = 1$ and $JN_2 = 1$.
Characteristic equation of this recursive relation is $t^3 - t^2 - 2 = 0$. From the Cardano´s formula for the cubic equation we can see that the charateristic equation of the Jacobsthal-Narayana sequence has one real root α and two distict complex roots β, γ. From[11] we have the following relation about the Binet formula of Jacobsthal-Narayana sequence JN_r.

$$JN_r = \frac{\alpha^{r+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{r+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{r+1}}{(\gamma-\alpha)(\gamma-\beta)}. \tag{6}$$

Also, the generating function of Jacobsthal-Narayana sequence is

$$\sum_{r=0}^{\infty} JN_r x^n = \frac{x}{1-x-2x^3}. \tag{7}$$

2 Difference Sequence of Jacobsthal-Narayana Numbers

In this section we introduce difference sequence of Jacobsthal-Narayana numbers and invsetigate some algebraic properties of this sequence.

Definition 2.1. Let JN_r denotes the r^{th} term of Jacobsthal-Narayana numbers. Then, difference sequence of Jacobsthal-Narayana numbers is denoted by $\Delta(JN_r)$ and is defined by

$$\Delta(JN_r) = JN_{r+1} - JN_r. \tag{8}$$

Theorem 2.2. *Let JN_r denotes the r^{th} term of Jacobsthal-Narayana numbers. Then the first difference sequence of Jacobsthal-Narayana numbers is given by*

$$\Delta(JN_r) = \left[\frac{\alpha-1}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^{r+1} + \left[\frac{\beta-1}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^{r+1} + \left[\frac{\gamma-1}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^{r+1}. \tag{9}$$

Proof. Exploiting the definition of difference sequence and Jacobsthal-Narayana numbers, we obtain

$$\Delta(JN_r) = JN_{r+1} - JN_r = \frac{\alpha^{r+2}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{r+2}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{r+2}}{(\gamma-\alpha)(\gamma-\beta)}$$
$$- \left[\frac{\alpha^{r+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{r+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{r+1}}{(\gamma-\alpha)(\gamma-\beta)}\right]$$
$$= \frac{(\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r+1} + \frac{(\beta-1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{r+1} + \frac{(\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r+1}.$$

□

By similar method, one can prove the following theorem about the second difference sequence of Jacobsthal-Narayana numbers.

Theorem 2.3. *Let JN_r denotes the r^{th} term of Jacobsthal Narayana numbers. Then the second difference sequence of Jacobsthal-Narayana numbers is given by*

$$\Delta^2(JN_r) = \left[\frac{(\alpha-1)^2}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^{r+1} + \left[\frac{(\beta-1)^2}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^{r+1} + \left[\frac{(\gamma-1)^2}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^{r+1}. \tag{10}$$

Now, we give the j^{th} difference sequence of Jacobsthal-Narayana numbers.

Theorem 2.4. *Let JN_r denotes the r^{th} term of Jacobsthal Narayana numbers and j is a natural number. Then, the j^{th} difference sequence of Jacobsthal-Narayana numbers is given by*

$$\Delta^j(JN_r) = \left[\frac{(\alpha-1)^j}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^{r+1} + \left[\frac{(\beta-1)^j}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^{r+1} + \left[\frac{(\gamma-1)^j}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^{r+1}. \tag{11}$$

Proof. We prove this theorem by mathematical iduction on j.
For $j = 1$ the result is true according to theorem (2.2).
Now, suppose that the result is true for $j = k$. In exact, suppose that

$$\Delta^k(JN_r) = \frac{(\alpha-1)^k}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r+1} + \frac{(\beta-1)^k}{(\beta-\alpha)(\beta-\gamma)}\beta^{r+1} + \frac{(\gamma-1)^k}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r+1}.$$

We prove that the result is true for $j = k+1$. By definition of difference sequence for $j = k+1$ we get

$$\Delta^{k+1}(JN_r) = \Delta^k(JN_{r+1}) - \Delta^k(JN_r)$$
$$= \left[\tfrac{(\alpha-1)^k}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r+2} + \tfrac{(\beta-1)^k}{(\beta-\alpha)(\beta-\gamma)}\beta^{r+2} + \tfrac{(\gamma-1)^k}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r+2}\right] - \left[\tfrac{(\alpha-1)^k}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r+1} + \tfrac{(\beta-1)^k}{(\beta-\alpha)(\beta-\gamma)}\beta^{r+1} + \tfrac{(\gamma-1)^k}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r+1}\right]$$
$$= \left[\tfrac{(\alpha-1)^k\alpha^{r+1}}{(\alpha-\beta)(\alpha-\gamma)}\right](\alpha-1) + \left[\tfrac{(\beta-1)^k\beta^{r+1}}{(\beta-\alpha)(\beta-\gamma)}\right](\beta-1) + \left[\tfrac{(\gamma-1)^k\gamma^{r+1}}{(\gamma-\alpha)(\gamma-\beta)}\right](\gamma-1)$$
$$= \tfrac{(\alpha-1)^{k+1}}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r+1} + \tfrac{(\beta-1)^{k+1}}{(\beta-\alpha)(\beta-\gamma)}\beta^{r+1} + \tfrac{(\gamma-1)^{k+1}}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r+1}.$$

Hence, the result is true for $j = k+1$. Consequently, by mathematical induction, the result is true for every natural number j. □

3 Jacobsthal-Narayana difference sequence

In this section, by using the notion of different sequence, we introduce the new sequence, namely, Jacobsthal-Narayana difference sequence. Then, we study some properties of this sequence and present the Binet formula and generating function of Jacobsthal-Narayana difference sequence.

Definition 3.1. Let r, n are natural numbers and $\{JN_r^{(i)}\}_{r=0}^{\infty}$ denotes the Jacobsthal-Naryana difference sequence. This sequence is defined by the following relation

$$\{JN_r^{(i)}\} = \{\Delta^i(JN_r)\}. \tag{12}$$

Remark 3.2. For $j = 0, 1$ the values of Jacobsthal-Narayana difference sequence are given by:

$$\{JN_r^{(0)}\} = \{\Delta^0(JN_r)\} = \{JN_r\} = \{JN_{r-1} + 2JN_{r-3}\} = \{0, 1, 1, 1, 3, 5, 7, 13, 23, 37, 63, 109, ...\} .$$

$$\begin{aligned}\{JN_r^{(1)}\} &= \{\Delta^1(JN_r)\} = \{\Delta(JN_r)\} = \{JN_{r+1} - JN_r\} \\ &= \{\left[\frac{\alpha - 1}{(\alpha-\beta)(\alpha-\gamma)}\right]\alpha^{r+1} + \left[\frac{\beta - 1}{(\beta-\alpha)(\beta-\gamma)}\right]\beta^{r+1} + \left[\frac{\gamma - 1}{(\gamma-\alpha)(\gamma-\beta)}\right]\gamma^{r+1}\}_{r=0}^{\infty},\end{aligned}$$

where α, β and γ are the roots of characteristic equation $t^3 - t^2 - 2 = 0$.

Next theorem shows that the Jacobsthal-Narayana difference sequence has similar recursive relation (in comparision) with Jacobsthal-Narayana sequence.

Theorem 3.3. *The Jacobsthal-Narayana difference sequence satisfies the following recursive relation*

$$JN_r^{(j)} = JN_{r-1}^{(j)} + 2JN_{r-3}^{(j)}. \tag{13}$$

Proof. We prove this theorem by mathematical induction on j. For $j = 1$ we have

$$JN_r^{(1)} = \Delta^1(JN_r) = \Delta(JN_r) = JN_{r+1} - JN_r.$$

By definition of Jacobsthal-Narayana sequence(JN_r) we get

$$\begin{aligned}JN_r^{(1)} &= (JN_r + 2JN_{r-2}) - (JN_{r-1} + JN_{r-3}) = (JN_r - JN_{r-1}) + 2(JN_{r-2} - JN_{r-3}) \\ &= \Delta(JN_{r-1}) + 2\Delta(JN_{r-3}) = JN_{r-1}^{(1)} + 2JN_{r-3}^{(1)}.\end{aligned}$$

Thus the result is true for $j = 1$.
Now suppose that the result is true for $j = k$. In exact, suppose that

$$JN_r^{(k)} = JN_{r-1}^{(k)} + 2JN_{r-3}^{(k)}. \tag{14}$$

For $j = k + 1$, we get

$$\begin{aligned}JN_r^{(k+1)} &= \Delta^{k+1}(JN_r) = \Delta^k(JN_{r+1}) - \Delta^k(JN_r) = JN_{r+1}^{(k)} - JN_r^{(k)}. \\ &= JN_r^{(k)} + 2JN_{r-2}^{(k)} - (JN_{r-1}^{(k)} + 2JN_{r-3}^{(k)}) = (JN_r^{(k)} - JN_{r-1}^{(k)}) + 2(JN_{r-2}^{(k)} - JN_{r-3}^{(k)}) \\ &= \Delta^{k+1}(JN_{r-1}) + 2\Delta^{k+1}(JN_{r-3}) = JN_{r-1}^{(k+1)} + 2JN_{r-3}^{(k+1)}.\end{aligned}$$

Thus the result is true for $j = k + 1$. Consequently the result is true for every j.

□

Following theorem presents the generating function of Jacobsthal-Narayana difference sequence.

Theorem 3.4. *The generating function of Jacobsthal-Narayana difference sequence* $(JN_r^{(j)})$ *is given by*

$$g^{(j)}(x) = \frac{JN_0^{(j)} + (JN_1^{(j)} - JN_0^{(j)})x + (JN_2^{(j)} - JN_1^{(j)})x^2}{1 - x - 2x^3} \tag{15}$$

Proof. Suppose that the generating function of Jacobsthal-Narayana difference sequence has the the formal power series $g^{(j)}(x) = \sum_{r=0}^{\infty} JN_r^{(j)} x^r = JN_0^{(j)} + JN_1^{(j)} x + JN_2^{(j)} x^2 + JN_3^{(j)} x^3 + \cdots$.
Then, we obtain

$$g^{(j)}(x) - xg^{(j)}(x) - 2x^3 g^{(j)}(x) = JN_0^{(j)} + (JN_1^{(j)} - JN_0^{(j)})x + (JN_2^{(j)} - JN_1^{(j)})x^2 + (JN_3^{(j)} - JN_2^{(j)} - 2JN_0^{(j)})x^3 + \cdots .$$

According to theorem(3.3), we know that $JN_r^{(j)} - JN_{r-1}^{(j)} - 2JN_{r-3}^{(j)} = 0$. Therefore, on the right side of the previous equality, the third and subsequent sentences are all equal to zero. Thus, we get

$$g^{(j)}(x)(1 - x - 2x^3) = JN_0^{(j)} + (JN_1^{(j)} - JN_0^{(j)})x + (JN_2^{(j)} - JN_1^{(j)})x^2.$$

Consequently, we obtain

$$g^{(j)}(x) = \frac{JN_0^{(j)} + (JN_1^{(j)} - JN_0^{(j)})x + (JN_2^{(j)} - JN_1^{(j)})x^2}{1 - x - 2x^3}.$$

□

Now, we give the Binet Formula of Jacobsthal-Narayana difference sequence $(JN_r^{(j)})$.

Theorem 3.5. *The Binet formula of Jacobsthal-Narayana difference sequence* $(JN_r^{(j)})$ *is given by*

$$JN_r^{(j)} = \frac{k_1 \alpha^r}{(\alpha - \beta)(\alpha - \gamma)} + \frac{k_2 \beta^r}{(\beta - \alpha)(\beta - \gamma)} + \frac{k_3 \gamma^r}{(\gamma - \alpha)(\gamma - \beta)}, \tag{16}$$

where,
$k_1 = JN_0^{(j)} \alpha^2 + (JN_1^{(j)} - JN_0^{(j)})\alpha + (JN_2^{(j)} - JN_1^{(j)})$,
$k_2 = JN_0^{(j)} \beta^2 + (JN_1^{(j)} - JN_0^{(j)})\beta + (JN_2^{(j)} - JN_1^{(j)})$,
$k_3 = JN_0^{(j)} \gamma^2 + (JN_1^{(j)} - JN_0^{(j)})\gamma + (JN_2^{(j)} - JN_1^{(j)})$.

Proof. By theorem(3.3), we see that the characterzaion equation of Jacobsthal-Narayana difference sequence is $f(x) = x^3 - x^2 - 2 = 0$. This equation has three distict roots α, β and γ. So, $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ are the roots of equation $h(x) = f(\frac{1}{x}) = 1 - x - 2x^3 = 0$. Hence, we get

$$h(x) = 1 - x - 2x^3 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x). \tag{17}$$

Therefore, using the generation function of Jacobsthal-Narayana difference sequence we have

$$g^{(j)}(x) = \sum_{r=0}^{\infty} JN_r^{(j)} x^r = \frac{JN_0^{(j)} + (JN_1^{(j)} - JN_0^{(j)})x + (JN_2^{(j)} - JN_1^{(j)})x^2}{1 - x - 2x^3} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x} + \frac{C}{1 - \gamma x} \tag{18}$$

$$= A \sum_{r=0}^{\infty} (\alpha x)^r + B \sum_{r=0}^{\infty} (\beta x)^r + \sum_{r=0}^{\infty} (\gamma x)^r = \sum_{r=0}^{\infty} \left[A\alpha^r + B\beta^r + C\gamma^r \right] x^r. \tag{19}$$

Thus, we obtain

$$g^{(j)}(x) = \frac{A(1-\beta x)(1-\gamma x) + B(1-\alpha x)(1-\gamma x) + C(1-\alpha x)(1-\beta x)}{(1-\alpha x)(1-\beta x)(1-\gamma x)}. \tag{20}$$

Hence, according to relations (17), (18), we deduce that

$$JN_0^{(j)} + (JN_1^{(j)} - JN_0^{(j)})x + (JN_2^{(j)} - JN_1^{(j)})x^2 = A(1-\beta x)(1-\gamma x) + B(1-\alpha x)(1-\gamma x) + C(1-\alpha x)(1-\beta x). \tag{21}$$

Substituting $x = \frac{1}{\alpha}$, and by some computation, we get

$$A = \frac{JN_0^{(j)}\alpha^2 + (JN_1^{(j)} - JN_0^{(j)})\alpha + (JN_2^{(j)} - JN_1^{(j)})}{(\alpha-\beta)(\alpha-\gamma)}. \tag{22}$$

Similarly, we get

$$B = \frac{JN_0^{(j)}\beta^2 + (JN_1^{(j)} - JN_0^{(j)})\beta + (JN_2^{(j)} - JN_1^{(j)})}{(\beta-\alpha)(\beta-\gamma)}, C = \frac{JN_0^{(j)}\gamma^2 + (JN_1^{(j)} - JN_0^{(j)})\gamma + (JN_2^{(j)} - JN_1^{(j)})}{(\gamma-\alpha)(\gamma-\beta)}. \tag{23}$$

If we put the values of A, B and C in (19), we deduce that

$$JN_r^{(j)} = \frac{k_1\alpha^r}{(\alpha-\beta)(\alpha-\gamma)} + \frac{k_2\beta^r}{(\beta-\alpha)(\beta-\gamma)} + \frac{k_3\gamma^r}{(\gamma-\alpha)(\gamma-\beta)}, \tag{24}$$

where,
$k_1 = JN_0^{(j)}\alpha^2 + (JN_1^{(j)} - JN_0^{(j)})\alpha + (JN_2^{(j)} - JN_1^{(j)})$,
$k_2 = JN_0^{(j)}\beta^2 + (JN_1^{(j)} - JN_0^{(j)})\beta + (JN_2^{(j)} - JN_1^{(j)})$,
$k_3 = JN_0^{(j)}\gamma^2 + (JN_1^{(j)} - JN_0^{(j)})\gamma + (JN_2^{(j)} - JN_1^{(j)})$. □

4 Conclusion

In this paper, we considered the Jacobsthal-Narayana numbers. Then, we introduced a new difference sequence by using the difference relation to the Jacobsthal-Narayana numbers, namely, Jacobsthal-Narayana difference sequence. We studied some algebraic properties of this sequence. Then we represented the Binet formula and generating functions related to this sequence. For the future works, one can combine the concept of difference sequence to other sequences and get new results about them.

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