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Mixed double Roman domination number of a graph

Mina Valinavaz[1]
Azarbaijan Shahid Madani University Tabriz, I.R. Iran

Abstract

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. A*mixed double Roman dominating function* ($MDRDF$) of G is a function $f : V \cup E \to \{0, 1, 2, 3\}$ satisfying the condition every element $x \in V \cup E$ for which $f(x) = 0$, is adjacent or incident to at least two elements $y, y' \in V \cup E$ for which $f(y) = f(y') = 2$ or one element $y'' \in V \cup E$ with $f(y'') = 3$, and if $f(x) = 1$, then element $x \in V \cup E$ must have at least one neighbor $y \in V \cup E$ with $f(y) \geq 2$. The mixed double Roman dominating number of G, denoted by $\gamma^*_{dR}(G)$. The weight of a $MDRDF$ f is $w(f) = \sum_{x \in V \cup E} f(x)$. The mixed double Roman domination number of G is the minimum weight of a mixed double Roman dominating function of G.

Keywords: Double Roman dominating function, Double Roman domination number, Mixed double Roman dominating function, Mixed double Roman domination number
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1 Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The minimum and maximum degree of a graph G are denote by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *Open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_v \in SN(v)$, and the *closed neighborhood*of S is the set $N[S] = N(S) \cup S$. For any $x \in V \cup E$, we denote by $N_m(x) = \{y \in V \cup E$: y is either adjacent or incident with $x\}$, and $N_m[x] = N_m(x) \cup \{x\}$. A *fan graph* $F_{1,n}$ is defined as the graph $K_1 + P_n$, where K_1 is the empty graph on one vertex and P_n is the path graph on n vertices. A set of vertices S in a graph G is *dominating set*of G if $N[S] = V$, that is, every vertex in $V \setminus S$ is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A more general version of domination, where each element $x \in V \cup E$ dominates $N_m[x]$, is *mixed domination*, see for examples [8, 11, 13]. For a mixed dominating set $S \subseteq V \cup E$, every element in denote $\gamma^*(G)$, of G is minimum cardinality of any mixed dominating set of G. A mixed dominating set is also called a

[1]speaker

total cover in [9, 10].

A *mixed Roman dominating functionMRDF* on a graph G is defined by Ahangar, Haynes and Tripodoro in [7] as a function $f : V \cup E \longrightarrow \{0, 1, 2\}$ satisfying the condition every element $x \in V \cup e$ for which $f(x) = 0$ is adjacent or incident to at least one element $y \in V \cup E$ for which $f(y) = 2$. The *weight*, $\omega(f)$, of f is defined as $f(V(G))$. The *mixed Roman domination number* of a graph G, denoted by $\gamma^*_R(G)$, is the minimum weight of any mixed Roman dominating function of G. (for example [2, 3]).

A *double Roman dominating function* on a graph G is defined by Beeler, Haynes and Hedetniemi in [7] as a function $f : V \longrightarrow \{0, 1, 2, 3\}$ having the property that if $f(u) = 0$, then vertex u has at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(u) = 1$, then vertex u must have at least one neighbor w with $f(w) \geq 2$. The *weight*, $\omega(f)$, of f is defined as $f(V(G))$. The *double Roman domination number* of a graph G, denoted by $\gamma_{dR}(G)$, is the minimum weight of any double Roman dominating function of G. Further results on the double Roman domination number can be found in [7, 1].

A *Edge double Roman dominating function*($EDRDF$) of graph G is defined by Valinavaz in [4, 5, 6]as function $f : E(G) \longrightarrow \{0, 1, 2, 3\}$ having the property that if $f(e) = 0$, then edge e has at least two neighbors assigned 2 under f or one neighbor e' with $f(e') = 3$, and if $f(e) = 1$, then edge e must have at least one neighbor e' with $f(e') \geq 2$. The *weight* of an edge double Roman dominating number of f, denote by $\omega(f)$, is the value $\sum_{e\in E(G)} f(e)$. The weight of a $EDRDF$, $\sum_{e\in E(G)} f(e)$. The minimum weight of a $EDRDF$ is the edge double roman domination number of G, denoted by $\gamma_{edR}(G)$.

We introduce the mixed version of double Roman domination as follows. Given a graph G, a *mixed double Roman dominating function* ($MDRDF$) of G is a function $f : V \cup E \to \{0, 1, 2, 3\}$ satisfying the condition every element $x \in V \cup E$ for which $f(x) = 0$, is adjacent or incident to at least two elements $y, y' \in V \cup E$ for which $f(y) = f(y') = 2$ or one element $y'' \in V \cup E$ with $f(y'') = 3$, and if $f(x) = 1$, then element $x \in V \cup E$ must have at least one neighbor $y \in V \cup E$ with $f(y) \geq 2$. The mixed double Roman dominating number of G, denoted by $\gamma^*_{dR}(G)$. The weight of a $MDRDF$ f is $w(f) = \sum\limits_{x\in V\cup E} f(x)$. The mixed double Roman domination number of G is the minimum weight of a mixed double Roman dominating function of G. A $MDRDF$ with minimum weight is called a γ^*_{dR}-function on G. Each $MDRDF$ determines a partition of the set $V \cup E = (V_0 \cup E_0) \cup (V_1 \cup E_1) \cup (V_2 \cup E_2) \cup (V_3 \cup E_3)$, where $V_i \cup E_i = \{x \in V \cup E : f(x) = i\}$. For the sake of simplicity, we will denote by $f[x] = f(N_m[x]) = \sum\limits_{v\in N_m[x]} f(v)$, for all $x \in V \cup E$. Let G be a graph. Suppose $T(G)$ is the graph whose vertex set is $V \cup E$ and two vertices in $T(G)$ are adjacent if and only if they are adjacent or incident in G. The proof of the following result is straightforward and therefore omitted.

Observation 1.1. For any graph G,

$$\gamma^*_{dR}(G) = \gamma_{dR}(T(G)) \text{ and } \gamma^*(G) = \gamma(T(G))$$

Prelem 1.2. For $n \geq 2$, $\gamma_{dR}(K_n) = 3$

Prelem 1.3. For a complete graph K_n with $n \geq 4$, $\gamma_{edR}(G)(K_n) = n$ if n is even, and $\gamma_{edR}(G)(K_n) = n + 1$ if n is odd.

2 Basic Properties

Proposition 2.1. *For any graph G, there exists a $\gamma^*_{dR}(G)$-function such that no edge and vertex needs to be assigned the value* 1.

Proof. Let f be a γ^*_{dR}-function on a graph G. Suppose that for some $x \in E \cup V$, $f(x) = 1$. This means that there is a element $x' \in N(x)$, such that either $f(x') = 2$ or $f(x') = 3$. If $f(x') = 3$, then we can achieve a mixed double Roman dominating function by reassigning a 0 to x. This results in a function with strictly less weight than f, contradicting that f is a γ^*_{dR}-function of G. If $f(x') = 2$, then we can create a mixed double Roman domination function g defined as follows: $g(x) = f(x)$ for all $x \notin \{x, x'\}, g(x) = 0$, and $g(x') = 3$. This result in a mixed double Roman domination function with weight equal to f. □

By Proposition 2.1, for any mixed double Roman dominating function f', there exists a mixed double Roman dominating function f no greater weight than f' for which $V_1 \cup E_1 = \emptyset$. Henceforth, without loss of generality, in determining the value $\gamma^*_{dR}(G)$ for any graph G, we can assume that $E_1 \cup V_1 = \emptyset$ for all mixed double Roman dominating functions under consideration.

Observation 2.2. Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2, V_3 \cup E_3)$ be a $MDRDF$ of a graph G. Then the following holds.

(a) Every element in $V_0 \cup E_0$ is dominated by some element of $V_3 \cup E_3$ or at least two elements of $V_2 \cup E_2$.

(b) $w(f) = 2|V_2 \cup E_2| + 3|V_3 \cup E_3|$.

(c) $V_2 \cup V_3 \cup E_2 \cup E_3$ is a mixed dominating set in G.

(d) It is not difficult to check that

$$\begin{aligned}\sum_{v\in V} f[v] + \sum_{e=uw\in E} f[e] &= \sum_{v\in V} f(N_m[v]) + \sum_{e=uw\in E} f(N_m[e]) \\ &= \sum_{v\in V} (2d(v)+1)f(v) \\ &+ \sum_{e=uw\in E} (d(u)+d(w)+1)f(uw).\end{aligned}$$

A classic result from [7] gives the following bounds on the double Roman dominating number of a graph G in terms of its domination number: $2\gamma(G) \le \gamma_{dR}(G) \le 3\gamma(G)$. We show that an analogous result applies for the mixed version as well.

Proposition 2.3. *For any graph G,*

$$2\gamma^*(G) \le \gamma^*_{dR}(G) \le 3\gamma^*(G)$$

Proof. For the lower bound, let $f = (V_0 \cup E_0, V_2 \cup E_2, V_3 \cup E_3)$ be a γ^*_{dR}-function of a graph G. Let $S \subseteq V \cup E$ be a $\gamma^*(G)$-set. Note that $(\emptyset, \emptyset, S)$ is a mixed double Roman dominating function. This yields the upper of $\gamma^*_{dR}(G) \le 3\gamma^*(G)$. On the other hand, by Observation 2.2(c), $V_2 \cup E_2 \cup V_3 \cup E_3$ is a mixed dominating set for G. Thus, $\gamma^*(G) \le |V_2 \cup E_2| + |V_3 \cup E_3|$. We can obtain the lower bound,

$$\gamma^*_{dR}(G) = 2|V_2 \cup E_2| + 3|V_3 \cup E_3| \geq 2(|V_2 \cup E_2| + |V_3 \cup E_3|) \geq 2\gamma^*(G).$$

□

Both the bounds of Proposition 2.3 are sharp. For the upper bound, as we have seen, the family of non-trivial stars $K_{1,n-1}$ has $\gamma^*(K_{1,n-1}) = 1$ and $\gamma^*_{dR}(K_{1,n-1}) = 3$. For the lower bound, we also recall the empty graph $\overline{K_n}$, has $\gamma^*(\overline{K_n}) = 1$ and $\gamma^*_{dR}(\overline{K_n}) = 2$ A graph G is said to be a double Roman graph if $\gamma_{dR}(G) = 3\gamma(G)$. Similarly, we say that a graph G is a *mixed double Roman graph* if $\gamma^*_{dR}(G) = 3\gamma^*(G)$.

Proposition 2.4. *A graph G is mixed double Roman graph if and only if it has a γ^*_{dR}-function $f = (V_0 \cup E_0, V_2 \cup E_2, V_3 \cup E_3)$ with $|V_2 \cup E_2| = 0$.*

Proof. Let f be a γ^*_{dR}-function on G with $|V_2 \cup E_2| = 0$. Taking into account that $V_2 \cup E_2 \cup V_3 \cup E_3 = V_3 \cup E_3$ is a mixed dominating set in G and that $\gamma^*_{dR}(G) = w(f) = 2|V_2 \cup E_2| + 3|V_3 \cup E_3| = 3|V_3 \cup E_3|$, it is derived that $\gamma_{dR}(G) = 3\gamma(G)$. Thus, G is a mixed double Roman graph.
Conversely, assume that G is a mixed double Roman graph, that is, $\gamma_{dR}(G) = 3\gamma(G)$. Let $X \subseteq V \cup E$ be a mixed dominating set in G and define a $MDRDF$ f as follows: $f(X) = 3$ for every $x \in X$ and $f(x) = 0$, otherwise. Clearly, f is a γ^*_{dR}-function on G such that $|V_2 \cup E_2| = 0$. □

We observe that $\gamma^*_{dR}(G) = 2$ if and only if G is the trivial graph K_1. We conclude the section by considering graphs having small mixed double Roman domination numbers.

Proposition 2.5. *Let G be a connected graph of order $n \geq 2$. Then*

1. *$\gamma^*_{dR}(G) = 3$ if and only if $G = K_{1,n-1}$.*

2. *$\gamma^*_{dR}(G) = 4$ if and only if $G = \{\overline{K_2}, K_3\}$.*

3. *$\gamma^*_{dR}(G) = 5$ if and only if $G = \{K_{1,n-2} \cup K_1, K_{1,n-1} + e\}$.*

Proof. Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2, V_3 \cup E_3)$ be a γ^*_{dR}-function of G such that $V_1 \cup E_1 = \emptyset$(by Proposition 2.1).

1. If $G \in K_{1,n-1}$, then clearly $\gamma^*_{dR}(G) = 3$. Conversely, let $\gamma^*_{dR}(G) = 3$, it follows that $|V_3 \cup E_3| = 1$ and $V_2 \cup E_2 = \emptyset$. In the former case, we deduce that $G = K_{1,n-1}$.

2. If $\gamma^*_{dR}(G) = 4$, then $|V_2 \cup E_2| = 2$ and $V_3 \cup E_3 = \emptyset$. Thus, we may assume that $V_2 \cup E_2 = \{x, y\}$. If G is disconnected, then $G \in \overline{K_2}$. Hence, we may assume that G is connected. Then x, y dominated all the elements of $(V \cup E)$, implying that $y \in V_2$ and $x \in E_2$. It follows that $G = K_3$. The converse is obvious.

3. Assume that $\gamma^*_{dR}(G) = 5$. We deduce from $2|V_2 \cup E_2| + 3|V_3 \cup E_3| = 5$ that $|V_2 \cup E_2| = |V_3 \cup E_3| = 1$. Thus, we may assume that $V_2 \cup E_2 = \{x\}$ and $V_3 \cup E_3 = \{y\}$. If G is disconnected, then $G \in K_{1,n-2} \cup K_1$ for $n \geq 3$ and the result holds. Hence, we may assume that G is connected. Then y dominated all the elements of $(V \cup E) \backslash \{x\}$, implying that $y \in V_3$ and $x \in E_2$. It follows that $G = K_{1,n-1} + e$ for $n \geq 3$. The converse is obvious.

□

3 Bounds On the Mixed Double Roman Domination Number

Lemma 3.1. *Let f be a γ^*_{dR}-function on a graph G. If $f[e] = f(N_m[e]) = 2$ for some edge $e = uv$, then there exist at least $d(u) + d(v) - 2 \geq 2$ edges e' for which $f[e'] \geq 4$.*

Proof. Let $e = xy$ be an edge satisfying the conditions of the lemma. As $f[e] = 2$, we may deduce that $f(xy) = 2$ and that $f(x) = f(y) = f(xx') = f(yy') = 0$, for all $x' \in N(x)$ and $y' \in N(y)$. Since f is a $MDRDF$, it follows that for f to dominate x(respectively, y), $d(x) \geq 2$ (respectively, $d(y) \geq 2$). Since $f(xx') = 0$ for all $x' \in N(x)$, either $f(x') = 2$ or there exists a vertex $t \in N(x')\backslash\{x\}$ for which $f(x't) = 3$ to dominate xx'. In any case, it is derived that $f[xx'] \geq f(xy) + f(x't) \geq 4$ for all $x' \in N(x)\backslash\{y\}$. Reasoning analogously, we may conclude that $f[yx'] \geq f(xy) + f(x't) \geq 4$ for all $x' \in N(y)\backslash\{x\}$. Summing up, there are at least $d(x) + d(y) - 2 \geq 2$ edges e' such that $f[e'] \geq 4$. □

Lemma 3.2. *Let f be a γ^*_{dR}-function on a graph G with no isolated vertices. If $f[v] = 2$ for some $v \in V$, then $f[v'] \geq 4$ for all $v' \in N(v)$.*

Proof. Assume that $f[v] = 2$. Thus, $f(v) = 2$ and $f(x) = 0$ for all $x \in N_m(v)$. Since G has no isolated vertices, $d(v) \geq 1$. Let $v' \in N(v)$. Since f is a $MDRDF$, there is an element $x'_v \in N(m)(v')$ with $f(xv') = 3$. Therefore, $f[v'] \geq f(v') + f(v) + f(x'_v) = 0 + 2 + 3 = 5$. □

Next we give a lower bound on the mixed double Roman domination number of a graph in terms of its order, size, and maximum degree.

Proposition 3.3. *Let G be a graph of order n, size m, and maximum degree $\Delta \geq \delta \geq 1$. Then*

$$\gamma^*_{dR}(G) \geq \lceil \tfrac{3(m+n)}{2\Delta+1} \rceil.$$

Proof. Let f be a $\gamma^*_{dR} - MDRDF$ in G. Note that for any element $x \in V_0 \cup E_0 \cup V_2 \cup E_2 \cup V_3 \cup E_3$, we have that $f[x] \geq 3$. Combining this with Observation 2.2, Lemma 3.1and Lemma 3.2, we obtain that $\sum\limits_{e\in E} f[e] \geq 3|E| = 3m$ and $\sum\limits_{v\in V} f[v] \geq 3|V| = 3n$. Therefore,

$$\begin{aligned} 3(m+n) &\leq \sum_{v\in V} f[v] + \sum_{e=uv\in E} f[uv] \\ &= \sum_{v\in V} (2d(v)+1)f(v) + \sum_{e=uv\in E} (d(u)+d(v)+1)f(uv) \\ &\leq (2\Delta+1)\left(\sum_{v\in V} f(v) + \sum_{e=uv\in E} f(uv)\right) \\ &= (2\Delta+1)\gamma^*_{dR}(G). \end{aligned}$$

which concludes the proof □

Corollary 3.4. *If G is an* r-regular *graph of order n, then*

$$\gamma^*_{dR}(G) \leq \lceil \tfrac{3n(r+2)}{2(2r+1)} \rceil.$$

Corollary 3.5. *If G is a* cubic graph *of order n, then*

$$\gamma^*_{dR}(G) \geq \lceil \tfrac{15n}{14} \rceil.$$

As can be seen in our next couple of results, the bound of Proposition 3.3 is sharp for paths P_n where $n \equiv 0, 3 \pmod 5$ and cycles C_n where $n \equiv 0, 2 \pmod 5$. Hence, the bound of Corollary 3.4 is sharp for these cycles as well.

Proposition 3.6. *For $n \geq 2$,*

$$\gamma^*_{dR}(P_n) = \begin{cases} \lceil \frac{6n-3}{5} \rceil & \text{if} \quad n \equiv 0, 3 \pmod 5 \\ \lceil \frac{6n-3}{5} \rceil + 1 & \text{if} \quad n \equiv 1, 2, 4 \pmod 5 \end{cases}$$

Proof. Assume that $P_n = v_1 v_2 .... v_{5\lfloor \frac{n}{5} \rfloor + j} (0 \leq j \leq 4)$ is a path on n vertices and $Z = V(P_n) \cup E(P_n)$. Note that $\gamma^*_{dR}(P_2) = \gamma^*_{dR}(P_3) = 3$ and $\gamma^*_{dR}(P_4) = 6$. Assume that $n \geq 5$. Define $f : Z \to \{0, 2, 3\}$ by $f(v_{5i-3}) = 3$ and $f(v_{5i-1}v_{5i}) = 3$ for $1 \leq i \leq \lfloor \frac{n}{5} \rfloor$ and $f(Z) = 0$ otherwise if $n \equiv 0(\mod n)$ and $f(v_n) = 2$ if $n \equiv 1 \pmod n$. Now assume that $n \equiv 2, 3, 4(\mod n)$, than $f(v_{5i-3}) = 3$ for $1 \leq i \leq \lceil \frac{n}{5} \rceil$ and $f(v_{5i-1}v_{5i}) = 3$ for $1 \leq i \leq \lfloor \frac{n}{5} \rfloor$, and $f(vn - 1v_n) = 3$ and $f(Z) = 0$ otherwise. It is easy to see that f is a $MDRDF$ of P_n of weight $\lceil \frac{6n-3}{5} \rceil$ if $n \equiv 0, 3 \pmod n$ and $w(f) = \lceil \frac{6n-3}{5} \rceil + 1$ if $n \equiv 1, 2, 4 \pmod n$. Therefore,

$$\gamma^*_{dR}(P_n) \leq \begin{cases} \lceil \frac{6n-3}{5} \rceil & \text{if} \quad n \equiv 0, 3 \pmod 5 \\ \lceil \frac{6n-3}{5} \rceil + 1 & \text{if} \quad n \equiv 1, 2, 4 \pmod 5 \end{cases}$$

To prove the lower bound, let f be a $MDRDF$. Since at least three elements from $V \cup E$ are required to dominate any five consecutive vertices on a path, and these three elements dominate at most 5 consecutive edges, it is straightforward to check that $\gamma^*_{dR}(P_n)$ is at least $3\lceil \frac{n}{5} \rceil$ if $n \equiv 0, 2, 3, 4 \pmod 5$ and is at least $3\lceil \frac{n}{5} \rceil + 2$ if $n \equiv 1 \pmod 5$. Simplifying, we have that $\gamma^*_{dR}(P_n)$ is bounded below by $\lceil \frac{6n-3}{5} \rceil$ if $n \equiv 0, 3 \pmod n$ and $\lceil \frac{6n-3}{5} \rceil + 1$ if $n \equiv 1, 2, 4 \pmod n$, the result holds. □

Proposition 3.7. *For $n \geq 3$,*

$$\gamma^*_{dR}(C_n) = \begin{cases} \lceil \frac{6n}{5} \rceil & \text{if} \quad n \equiv 0, 2 \pmod 5 \\ \lceil \frac{6n}{5} \rceil + 1 & \text{if} \quad n \equiv 1, 3, 4 \pmod 5 \end{cases}$$

Proof. Note that $\gamma^*_{dR}(C_3) = 5$ and $\gamma^*_{dR}(C_4) = 6$. Assume that $n \geq 5$. Applying Proposition 3.3, we have

$$\gamma^*_{dR}(C_n) \geq \lceil \frac{3(n+n)}{2\Delta+1} \rceil = \lceil \frac{6n}{5} \rceil.$$

Using an argument similar to the one for paths, we note that this lower bound on $\gamma^*_{dR}(C_n)$ is strict when $n \equiv 1, 3, 4 \pmod 5$.

To prove the upper bound, we define a $MDRDF$ on C_n, let $V(C_n) = \{v_1, v_2, ..., v_{5\lfloor \frac{n}{5} \rfloor + j}\}$ be the set of vertices of C_n, where $0 \leq j \leq 4$. Consider the function in G defined as follows: If $n \equiv 0, 4 \pmod 5$, then $f(v_{5i-3}) = 3$ and $f(v_{5i-1 v_5 i}) = 3$ for $1 \leq i \leq \lceil \frac{n}{5} \rceil$, $f(x) = 0$ otherwise and $f(v_n) = 3$ if $n \equiv 1, 2 \pmod 5$, and $f(v_n v_1) = 2$ if $n \equiv 3 \pmod 5$. Since f is $MDRDF$ with $w(f) = \lceil \frac{6n}{5} \rceil + 1$ if $n \equiv 1, 3, 4 \pmod 5$ and $w(f) = \lceil \frac{6n}{5} \rceil$ otherwise, the result holds. □

We note that the bound given by Corollary 3.5 is also sharp. To illustrate this, we construct a family $\mathcal{A}$ of cubic graphs with order $7t$ for any even integer $t \geq 2$ as follows: Let F_t be the union of t claws $K_{1,3}$ where each claw has center v_i for $1 \leq i \leq t$, and let H_t be the union of $\frac{3t}{2}$ edges. Construct a graph G from $F_t \cup H_t$ by adding $6k$ new edges, each joining a vertex in F_t to a vertex in H_t, in such a way that the resulting graph is cubic Note that each of the additional $6t$ edges is dominated by the edges of H_t. Thus, the set $S = E(H_t) \cup \{v_i | 1 \leq i \leq t\}$ is a mixed double Roman dominating set of G, and assigning a 3 to each element of S and a 0 to all other elements of G yields a mixed double Roman dominating function with weight $3|S| = 3(\frac{3t}{2} + t) = \frac{15t}{2} = \frac{15(7t)}{2\times 7} = \frac{15n}{14}$. For a example where t=3, see Fig 1.

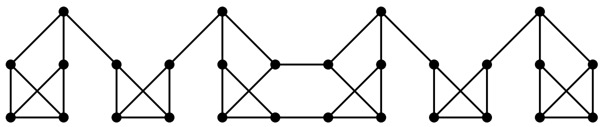

Figure 1: A cubic graph belonging to the family $\mathcal{A}$

Observation 3.8. For every connected graph G of order $n \geq 2$ and size m, $\gamma^*_{dR}(G) \leq 2(m+n) - 3$ with equality if and only if $G = K_2$.

Proof. Assume $xy \in E(G)$. Define $f : V(G) \cup E(G) \to \{0, 1, 2, 3\}$ by $f(x) = f(y) = 0, f(xy) = 3$ and $f(z) = 2$ for $z \in V(G) \cup E(G) - \{x, y, xy\}$ obviously f is a mixed double Roman dominating function of G and so $\gamma^*_{dR}(G) \leq 2m + 2n - 3$. If $G = K_2$, then clearly $\gamma^*_{dR}(G) = 3 = 2(m+n) - 3$. Let $\gamma^*_{dR}(G) = 2m+2n-3$, we show that $\Delta(G) = 1$. Suppose to the contrary that $\Delta(G) \geq 2$, let v be a vertex of maximum degree $\Delta(G)$ and $x_1, x_2 \in N(x)$. Then define the function $f : V(G) \cup E(G) \to \{0, 1, 2, 3\}$ by $f(x_1) = f(x_2) = 0, f(x) = 3$ and $f(z) = 2$ for $z \in V(G) \cup E(G) - \{x_1, x_2, x\}$. It is easy to see that f is an $MDRDF$ of G of weight $2(m - 2 + n - 3) + 3 = 2m + 2n - 7$ which is contradiction. Thus $\Delta(G) = 1$ and hence $G = K_2$. □

Proposition 3.9. *For every connected graph G of order n, size m and minimum degree $\delta(G) \geq 2$, $\gamma^*_{dR}(G) \leq 2n - 4 + \gamma_{edR}(G)$.*

Proof. Let f be a $\gamma_{edR}(G)$-function. Since $\gamma_{edR}(G) \leq \frac{5m}{4}$[**?**]. We deduce that $f(e) = 3$ for some edge $e = uv \in E(G)$. Define $g : V(G) \cup E(G) \to \{0, 1, 2, 3\}$ by $g(u) = g(v) = 0$, $g(x) = 2$ for $x \in V(G) - \{u, v\}$ and $g(x) = f(x)$ for $x \in E(G)$. It is easy to see that g is an $MDRDF$ of G and hence $\gamma^*_{dR} \leq w(g) = 2(n-2) + \gamma_{edR}(G)$. This completes the proof. □

Proposition 3.10. *For $1 \leq r \leq s$, $\gamma^*_{dR}(K_{r,s}) = 3r$.*

Proof. Let $X = \{x_1, x_2, ..., x_r\}$ and $Y = \{y_1, y_2, ..., y_s\}$ be the partite sets of $K_{r,s}$ with $1 \leq r \leq s$. Clearly, assigning a 3 to each vertex in X and 0 to each vertex in Y yields a $MDRDF$ of cardinality $3r$, so $\gamma^*_{dR}(K_{r,s}) \leq 3r$. We note that since X,Y are independent sets to dominate the edges of G, each edge must be assigned a 3 or must be incident to a vertex a assigned a 3. Thus, $\gamma^*_{dR}(K_{r,s}) \geq 3r$ and so $\gamma^*_{dR}(K_{r,s}) = 3r$. □

Proposition 3.11. *For any connected graph G,*

$$\max\{\gamma_{dR}(G), \gamma_{edR}(G)\} \leq \gamma^*_{dR}(G) \leq \gamma_{dR}(G) + \gamma_{edR}(G).$$

The lower bound is sharp for stars $K_{1,n}(n \geq 2)$ and the upper bound is sharp for fan graph.

Proof. If f is a $\gamma_{dR}(G)$-function and g is a $\gamma_{edR}(G)$- function, then the function $h : V \cup E \to \{0, 2, 3\}$ defined by $h(x) = f(x)$ for $x \in V$ and $h(x) = g(x)$ for $x \in E$, is clearly a mixed double Roman dominating function of G that implies $\gamma^*_{dR}(G) \leq \gamma_{dR}(G) + \gamma_{edR}(G)$.
To prove the lower bound, let f be a $\gamma^*_{dR}(G)$-function. First we show that $\gamma_{dR}(G) \leq \gamma^*_{dR}(G)$. Let $V(G) = \{v_1, v_2, ..., v_n\}$ and $E(v_i) = \{v_iv_j \in E(G)|i < j\}$. Define $g : V(G) \to \{0, 2, 3\}$ by $g(v_i) = f(v_i) \cup (\cup_{e \in E(v_i)} f(e))$ for each $1 \leq i \leq n$. Clearly, g is $DRDF$ of G of weight $w(f)$ that implies $\gamma_{dR}(G) \leq \gamma^*_{dR}(G)$. Now, we show that $\gamma_{edR}(G) \leq \gamma^*_{dR}(G)$. Suppose that $M = \{x_1y_1, x_2y_2, ..., x_ry_r\}$ is maximum matching in G and $X = \{w_1, w_2, ..., w_p\}$ is the set consisting of all M-unsaturated vertices. Let e_i be an edge incident to W_i for $1 \leq i \leq p$ and define $h : E(G) \to \{0, 2, 3\}$ by $h(x_iy_i) = f(x_iy_i) \cup f(x_i) \cup f(y_i)$ for $1 \leq i \leq r$, $h(e_j) = f(e_j) \cup f(w_j)$ for $1 \leq j \leq p$ and $h(e) = f(e)$ otherwise. Clearly, h is $EDRDF$ of G of weight $w(f)$ implying that $\gamma_{edR}(G) \leq \gamma^*_{dR}(G)$. This complete the proof. □

Proposition 3.12. *If G is a graph and $e \in E(\overline{G})$, then*

$$\gamma^*_{dR}(G) - 3 \leq \gamma^*_{dR}(G + e) \leq \gamma^*_{dR}(G) + 2.$$

Proof. To prove the upper bound, let f be a $\gamma^*_{dR}(G)$-function. Clearly, $g : V(G) \cup E(G) \cup \{e\} \to \{0, 1, 2, 3\}$ defined by $g(e) = 2$ and $g(x) = f(x)$ otherwise is a $MDRDF$ of $G + e$ and hence $\gamma^*_{dR}(G + e) \leq \gamma^*_{dR}(G) + 2$.
To prove the lower bound, assume that $e = vw$ and f is a $\gamma^*_{dR}(G+e)$-function. First let $f(e) = 0$. If $f(v) = f(w) = 0$ or $0 \notin \{f(v), f(w)\}$, then clearly the function f, restricted to G is a $MDRDF$ of G implying that $\gamma^*_{dR}(g) - 3 < \gamma^*_{dR}(G) \leq \gamma^*_{dR}(G + e)$. Assume, without loss of generality, that $f(w) = 0$ and $f(v) \neq 0$. Then the function $g : V(G) \cup E(G) \to \{0, 1, 2, 3\}$ defined by $g(w) = 2$ and $g(x) = f(x)$ otherwise, is a $MDRDF$ of G of weight $\gamma^*_{dR}(G + e) + 2$ and hence $\gamma^*_{dR}(g) - 3 \leq \gamma^*_{dR}(G + e)$. Now let $f(e) \neq 0$. Define $g : V(G) \cup E(G) \to \{0, 1, 2, 3\}$ by $g(w) = f(w) \cup f(e), g(v) = f(v) \cup f(e)$ and $g(x) = f(x)$ otherwise. It is to see that g is a $MDRDF$ of G of weight $\gamma^*_{dR}(G + e) + f(e)$ and so $\gamma^*_{dr}(g) - 3 \leq \gamma^*_{dR}(G) - f(e) \leq \gamma^*_{dR}(G + e)$. This completes the proof. □

Corollary 3.13. *For any edge e in a graph G,*

$$\gamma^*_{dr}(g) - 2 \leq \gamma^*_{dR}(G - e) \leq \gamma^*_{dR}(G) + 3.$$

Proposition 3.14. *For $n \geq 7$,*

$$\gamma^*_{dR}(K_n) = \begin{cases} n + 2 & \text{if} \quad n \equiv 1, 3 \pmod 4 \\ n + 3 & \text{if} \quad n \equiv 0, 2 \pmod 4 \end{cases}$$

Unless $n \leq 6$ in which cases $\gamma^*_{dR}(K_3) = 4, \gamma^*_{dR}(K_4) = 6, \gamma^*_{dR}(K_5) = 6, \gamma^*_{dR}(K_6) = 8$.

Proof. Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ be the set of vertices of C_n and $Z = V(K_n) \cup E(K_n)$. Assume that $n \leq 6$. Define $f : Z \to \{0, 2, 3\}$ by $f(v_1) = f(v_2v_3) = 3$ if $n = 4$ and $f(v_1) = f(v_2v_3) = 2$ for $n = 3$, $f(v_1) = f(v_2v_3) = f(v_4v_5) = 2$ if $n = 5$ and $f(v_1) = f(v_4) = f(v_2v_3) = f(v_5v_6) = 2$ if $n = 6$. Now assume that $n \geq 7$. By Proposition 1.2, 1.3

$$\gamma_{dR}(K_n) + \gamma_{edR}(K_{n-1}) \leq \begin{cases} n + 2 & \text{if} \quad n \equiv 1, 3 \pmod 4 \\ n + 3 & \text{if} \quad n \equiv 0, 2 \pmod 4 \end{cases}$$

□

Proposition 3.15. *Let G be a connected graph of order $n \geq 2$, size m and $\Delta(G) \geq 1$. Then* $\gamma^*_{dR}(G) \leq 2(m+n) - 4\Delta(G) + 1$.

Proof. The result holds from $G = K_2$. Thus, we may assume that $n \geq 3$ and $\Delta \geq 2$. Let v be a vertex of maximum degree $\Delta(G) = k \geq 2$. To simplify notation for a set $S \subseteq V \cup E$ of a graph G, let $N_m[S] = \bigcup_{v \in S} N_m[v]$, and define the function f_s by assigning 3 to every element of S, 0 every element in $N_m[S] \backslash S$ and 2 to all remaining elements in $V \cup E$, we note that f_v is a $MDRDF$ for any set $v \in V \cup E$. Then, $\gamma^*_{dR}(G) \leq w(f_v) = 2(m+n-2\Delta(G)-1)+3 = 2(m+n)-4\Delta(G)+1$ □

Proposition 3.10 shows that the bound of Proposition 3.15 is sharp. A set $S \subseteq V(G)$ is a *2-packing* set of G if $N[u] \cap N[v] = \emptyset$ holds for any two distinct vertices $u, v \in S$. The 2-packing number of G, denote $\rho(G)$, is defined as follow: $\rho(G) = \max\{|S| : S$ is a 2-packing set of $G\}$.

Observation 3.16. Let G be a connected graph of order $n \geq 2$ and size m. Then

$$\gamma^*_{dR}(G) \leq 2m + 2n - (4\delta(G) - 1)\rho(G).$$

Proof. Let $\{x_1, x_2, ..., x_k\}$ be a 2-packing of G. Define $f : V \cup E \to \{0, 1, 2, 3\}$ by $f(x_i) = 3, f(x) = 0$ for $x \in N(x_i)$ for $1 \leq i \leq k$ and $f(x) = 2$ otherwise. It is easy to see that f is an $MDRDF$ of G. Thus

$$\begin{aligned}
\gamma^*_{dR}(G) &\leq w(f) = 2(m+n-\sum_{i=1}^{k}(2deg(x_i)+1)+3k \\
&= 2m+2n-2\sum_{i=1}^{k}(2deg(x_i)+1)+3k \\
&\leq 2m+2n-2(2\delta(G)k+k)+3k \\
&\leq 2m+2n-(4\delta(G)-1)k.
\end{aligned}$$

□

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