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On the permanent of an even-dimensional $(0,1)$-polystochastic tensor of order n

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Abstract

In this paper, we prove that the permanent of a 4-dimensional polystochastic $(0,1)$-tensor of order n constructed using a special $n \times (n-1)$ row-Latin rectangle R with no transversals is positive. Also, we show that the permanent of an even-dimensional polystochastic $(0,1)$-tensor of order n constructed using the row-Latin rectangle R is positive. The result obtained here proves that each odd-dimensional Latin hypercube of order 4 has a transversal (Wanless' conjecture for odd-dimensional Latin hypercubes of order 4). We prove that the number of perfect matchings of the bipartite hypergraph associated to an even-dimensional polystochastic $(0,1)$-tensor of order 4 is positive.

Keywords: Latin hypercube, Permanent, Polystochastic tensor, Tensor, Transversal.

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1 Introduction

A *d-dimensional tensor of order* $n_1 \times \cdots \times n_d$, $\mathcal{A} = (a_{i_1\ldots i_d})_{n_1\times\cdots\times n_d}$ is a multi-array of entries $a_{i_1\ldots i_d} \in \mathbb{F}$, where $i_j = 1,\ldots,n_j$ for $j = 1,\ldots,d$ and $\mathbb{F}$ is a field. In this paper, we only consider real tensors, that is, those tensors for which $\mathbb{F} = \mathbb{R}$. When $n_1 = n_2 = \cdots = n_d = n$, we say that $\mathcal{A}$ is a square d-dimensional tensor of order n. The *diagonal* of a d-dimensional tensor of order $n_1 \times \cdots \times n_d$ is a selection of n_1 entries of $\mathcal{A}$ with the property that no two entries lie in the same hyperplane. The *permanent* of $\mathcal{A}$ is the sum of all diagonal products of $\mathcal{A}$. A tensor $\mathcal{A}$ is called *non-negative* if all its entries are at least zero. We refer the interested reader to [11] for more information on the theory of tensors.

Some properties of permanent were extended from matrices to tensors by Dow and Gibson [4]. Many difficult problems in other fields can be stated equivalently as those which ask for the permanents of certain associated tensors. Recently, Taranenko studied the applications of the permanents of tensors to combinatorial designs, the number of Steiner systems, transversals in Latin hypercubes, 1-factors of hypergraphs, and MDS codes [13]. Some other applications of the permanents can be found in [1], [8] and [10]. The problem of finding a maximum size matching in a general hypergraph is NP-hard. Also, the problem of finding a

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perfect matching in a 3-partite hypergraph is one of Karp's 21 NP-complete problems [6]. Although the permanent of a tensor is important in other fields, its calculation lies in the list of NP-hard problems [13]. To the best of our knowledge, no algorithm exists for the computation of the permanent of a tensor. For this reason, we propose an algorithm that allows us to compute the permanent of a tensor. This algorithm helps us to avoid manual computation.

This paper is organized as follows. In Section 2, we review some basic definitions together with the known properties of the permanent of a tensor. In Section 3, we first review the main definitions, and then examine the relation between the number of transversals of a d-dimensional Latin hypercube of order n and its corresponding $d+1$-dimensional polystochastic $(0,1)$-tensor of order n. Next, we prove that the permanent of a 4-dimensional polystochastic $(0,1)$-tensor of order n constructed using a special $n \times (n-1)$ row-Latin rectangle R with no transversals is positive. Also, we show that the permanent of an even-dimensional polystochastic $(0,1)$-tensor of order n constructed using the row-Latin rectangle R is positive. The result obtained here proves that each odd-dimensional Latin hypercube of order 4 has a transversal (Wanless' conjecture for odd-dimensional Latin hypercubes of order 4). In Section 4, we apply our main result to the theory of hypergraphs; we prove that the number of perfect matchings of the bipartite hypergraph associated to an even-dimensional polystochastic $(0,1)$-tensor of order 4 is positive. We conclude the paper with a brief conclusion in Section 5.

2 Preliminaries

In this section, the permanent of a tensor and some of its properties are discussed.

Definition 2.1 ([14]). Let $\mathcal{A}$ be a square d-dimensional tensor of order n. Given $k \in \{0, 1, \ldots, d\}$, a k-*dimensional plane* in $\mathcal{A}$ is a subtensor obtained by fixing $d-k$ indices and letting the other k indices vary from 1 to n. A 1-dimensional plane is said to be a *line*, and a $(d-1)$-dimensional plane is a *hyperplane*.

Definition 2.2 ([4]). Let $\mathcal{A}$ be a d-dimensional tensor of order $n_1 \times \cdots \times n_d$. The following sequence is called a *diagonal* of $\mathcal{A}$.

$$(a_{1\sigma_2(1)\ldots\sigma_d(1)}, a_{2\sigma_2(2)\ldots\sigma_d(2)}, \ldots, a_{n_1\sigma_2(n_1)\ldots\sigma_d(n_1)}),$$

where σ_k is one-to-one function from $\{1, \ldots, n_1\}$ to $\{1, \ldots, n_k\}$ for $k = 2, \ldots, d$.

Definition 2.3 ([4]). Let $\mathcal{A}$ be a d-dimensional tensor of order $n_1 \times \cdots \times n_d$. The *permanent* of $\mathcal{A}$ is defined by

$$per(\mathcal{A}) := \sum_{\sigma_k} \prod_{i=1}^{n_1} a_{i\sigma_2(i)\ldots\sigma_d(i)}, \tag{1}$$

where the summation runs over all one-to-one functions σ_k from $\{1, \ldots, n_1\}$ to $\{1, \ldots, n_k\}$ and $k = 2, \ldots, d$, with $per(\mathcal{A}) = 0$ if $n_1 > n_k$ for some k.

Definition 2.4. For a tensor $\mathcal{A} = (a_{i_1\ldots i_d})_{n_1\times\cdots\times n_d}$, the hyperplanes obtained by fixing $i_k, 1 \le k \le d$, are said to be *hyperplanes of type* k [4].

Example 2.5. If $\mathcal{A} = (a_{i_1i_2i_3})_{2\times2\times2}$, then

$$per(\mathcal{A}) = a_{111}a_{222} + a_{121}a_{212} + a_{211}a_{122} + a_{221}a_{112}.$$

The following property is similar to the Laplace expansion of the permanent of a matrix [9].

Proposition 2.6 ([13])**.** *If $\mathcal{A}$ is a d-dimensional tensor of order n, then*

$$per(\mathcal{A}) = \sum_{i_2,\dots,i_d=1}^{n} a_{1,i_2,\dots,i_d} per(\mathcal{A}(1|i_2|\dots|i_d)),$$

where $\mathcal{A}(1|i_2|\dots|i_d)$ is a d-dimensional tensor of order $n-1$, which is obtained from $\mathcal{A}$ by removing hyperplanes i_k of type k for $k = 1, i_2, \dots, i_d$.

3 The permanent of a $(0,1)$-polystochastic tensor

In this section, we first present some definitions that will be used in the proofs of our theorems. Next, we prove the aforementioned conjectures of Sun and Wanless in special case. To do so, we first prove Sun's conjecture in special case (a special case of Wanless' conjecture). Then, we prove Wanless' conjecture in special case by using induction. Also, we prove conjectures of Sun and Wanless for odd-dimensional Latin hypercubes of order 4.

Definition 3.1 ([13])**.** A non-negative tensor is *polystochastic* if the sum of the entries in each of its lines is equal to 1. A 2-dimensional polystochastic tensor is known as a *doubly stochastic matrix*.

Latin hypercubes generalize Latin squares to multidimensional arrays. The number of transversals in a Latin hypercube was described in [5].

Definition 3.2 ([13])**.** A d-dimensional *Latin hypercube* Q of order n is a d-dimensional tensor of order n with the property that every line contains pairwise distinct elements of the set $\{1, \dots, n\}$. A 2-dimensional Latin hypercube is called a *Latin square*, and a 3-dimensional Latin hypercube is known as a *Latin cube*.

Definition 3.3 ([14])**.** A *partial diagonal* p of length k in a d-dimensional tensor of order n is a set $\{\alpha^1, \dots, \alpha^k\}$ of k indices such that for any i and j, α^i and α^j are distinct in all components. A partial diagonal p is *positive* if all entries of $\mathcal{A}$ with indices in p are greater than 0.

Definition 3.4 ([13])**.** A *transversal* in a Latin hypercube Q is a diagonal that all elements are distinct.

There is a one-to-one correspondence between Latin hypercubes $Q = (q_{i_1 \dots i_d})_{n\times\dots\times n}$ and polystochastic $(0,1)$-tensors $\mathcal{A} = (a_{i_1\dots i_{d+1}})_{n\times\dots\times n}$, in the sense that $q_{i_1\dots i_d} = i_{d+1}$ if and only if $a_{i_1\dots i_{d+1}} = 1$.

The permanent of a polystochastic $(0,1)$-tensor is equal to the number of transversals in its corresponding Latin hypercube [13]. Latin hypercubes generalize Latin squares to multidimensional arrays. The number of transversals in a Latin hypercube was described in [5].

Definition 3.5 ([14])**.** A $k \times m$ *row-Latin rectangle* R is a table with k rows and m columns filled by m symbols in such a way that each row contains all the m symbols. A *transversal* in the rectangle R is the set of $\min\{k, m\}$ entries hitting each row, each column, and each symbol no more than once.

Definition 3.6 ([3])**.** We say that rectangles R and S are isotopic if S can be obtained by permuting the rows, columns, and symbols of R. The triple of permutations which achieves this is called an isotopism.

The interested reader is referred to [3] for the theory of row-Latin rectangles. In the following example, we present a row-Latin rectangle with no transversals.

Example 3.7. Let R be a row-Latin rectangle of the form below. We show that R has no transversals.

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1 \\ 1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1 \\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & 4 & \dots & \dots & n-3 & n-2 & n-1 \\ n-1 & 1 & 2 & 3 & \dots & \dots & n-4 & n-3 & n-2 \\ n-1 & 1 & 2 & 3 & \dots & \dots & n-4 & n-3 & n-2. \end{array}$$

In fact, suppose that R has a transversal $\mathtt{T}$. We may assume, without loss of generality, that $\mathtt{T}$ has a 1 in the first column. Then, it must have a 2 in the second column, ..., and an $n-2$ in the $(n-2)$th column. But, then there is no possible choice for the $(n-1)$th column, since all the rows that contain $n-1$ have been used. Thus, R has no transversals.

The following conjecture has been proposed by Sun [12] in 2008. It was proved by Taranenko in the special case $n=4$ [14].

Conjecture 3.8. Every 3-dimensional Latin hypercube of order n has a transversal.

To prove Theorem 3.11 below, we need the following lemma.

Lemma 3.9. *If A is a doubly stochastic $(0,1)$-matrix of order n that contains a positive partial diagonal of length $(n-2)$, then the partial diagonal can be extended to a positive partial diagonal of length $(n-1)$.*

Proof. This follows from the fact that, each doubly stochastic $(0,1)$-matrix is a permutation matrix. □

Lemma 3.9 is not valid for non-negative matrices of order $n \geq 5$. For instance, matrix A below has a positive diagonal of length 3, which cannot be extended to a positive partial diagonal of length 4.

$$A = \begin{bmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{2}{4} \\ 0 & \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ 0 & 0 & \frac{2}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 & 0 \\ \frac{2}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix}.$$

The following conjecture is equivalent to Sun's conjecture.

Conjecture 3.10. The permanent of a 4-dimensional polystochastic $(0,1)$-tensor of order n is positive.

In what follows, we use the special row-Latin rectangle R to present an algorithm for the study of Conjecture 3.10.

Theorem 3.11. *The permanent of a 4-dimensional polystochastic $(0,1)$-tensor of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive.*

Proof. We show that a 4-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0 cannot be constructed by using the special row-Latin rectangle R. To see this, we try to construct such a tensor, and we observe that this is not indeed possible. We exert the algorithm below step by step for the construction. (See Table 2.) This algorithm has five steps. In step 1, we define a doubly stochastic matrix B, and we select a positive diagonal for it. In step 2, we define the doubly stochastic matrices B_i and we select a positive diagonal for B_i, where $i=0,\dots,n-1$, using a row-Latin rectangle R with no transversals.

In step 4, we define the doubly stochastic matrices C_k, where $k = 1, \ldots, n-1$, and we extend a positive partial diagonal of length $n-2$ to a positive partial diagonal of length $n-1$ for C_k, where $k = 1, \ldots, n-1$. In step 5, we consider the vertical lines $(*, n-2, k, k)$, where $k = 1, \ldots, n-1$ and $* = 0, \ldots, n-1$, and we set equal to 1 the entries on these lines in order for $\mathcal{A}$ to remain polystochastic. In each step, we set equal to 1 some entries of $\mathcal{A}$, since $\mathcal{A}$ is a $(0,1)$-tensor by our hypothesis. At the end of step 5, we observe that it is not possible to construct a 4-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0.

Step 1: Consider the 2-dimensional plane B of the form

$$B = \mathcal{A}(*_1, *_2, 0, 0) \qquad *_1, *_2 = 0, \ldots, n-1.$$

Since $\mathcal{A}$ is a polystochastic tensor, B is a doubly stochastic matrix. Therefore, being a doubly stochastic matrix, B has a positive diagonal [9]. Without loss of generality, we set the entries of $\mathcal{A}$ with indices $(i, i, 0, 0)$ equal to 1, where $i = 0, 1, \ldots, n-1$, and we also consider these entries as a positive diagonal for B.

We denote these positive entries by 1_{s1} in Table 1. Notice that at the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step 2: Consider the 2-dimensional plane B_i of the form

$$B_i = \mathcal{A}(i, i, *_1, *_2), \qquad i = 0, \ldots, n-1, \qquad *_1, *_2 = 0, \ldots, n-1.$$

As before, B_i is a doubly stochastic matrix, and hence has a positive diagonal. To select a positive diagonal for B_i, we consider an $n \times (n-1)$ row-Latin rectangle R with no transversals. (Since a transversal in R gives a positive diagonal for $\mathcal{A}$, and our goal is to construct a polystochastic $(0,1)$-tensor with permanent equal to 0, we consider a row-Latin rectangle without any transversals.) We assume that the entries of $\mathcal{A}$ with indices $\{(i, i, \beta_i^j, \gamma_i^j)\}_{j=1}^n$ are equal to 1, and also form a positive diagonal for B_i containing $a_{i,i,0,0}$, where β_i^j and γ_i^j are determined according to the rectangle R as follows.

The entry in the $(i+1)$th row and the β_i^jth column of R is γ_i^j, where $i = 0, \ldots, n-1$.

In what follows, we introduce a row-Latin rectangle R, and we use it to choose β_i^j and γ_i^j. To do so, we consider R in the form below.

$$\begin{array}{cccccccc}
1 & 2 & 3 & 4 & \ldots & \ldots & n-3 & n-2 & n-1 \\
1 & 2 & 3 & 4 & \ldots & \ldots & n-3 & n-2 & n-1 \\
\vdots & \vdots & \vdots & \vdots & \ldots & \ldots & \vdots & \vdots & \vdots \\
1 & 2 & 3 & 4 & \ldots & \ldots & n-3 & n-2 & n-1 \\
n-1 & 1 & 2 & 3 & \ldots & \ldots & n-4 & n-3 & n-2 \\
n-1 & 1 & 2 & 3 & \ldots & \ldots & n-4 & n-3 & n-2.
\end{array}$$

We observed in Example 3.7 that R has no transversals. We consider a positive diagonal for B_i, where $i = 0, \ldots, n-1$, according to the row-Latin rectangle R and the discussion above, as follows.

For example, to choose a positive diagonal for B_0, we look at the first row of R. In this row, we see that the entry located in the first column is equal to 1 (that is, $R_{11} = 1$). Hence, we let $a_{0011} = 1$. Also, in this row of R, we see that $R_{1,n-1} = n-1$. Therefore, we let $a_{0,0,n-1,n-1} = 1$. Similarly, to select a positive diagonal for B_{n-1}, we look at the last row of R. In this row, we see that the entry located in the first column is equal to $n-1$ (that is, $R_{n1} = n-1$). Hence, we let $a_{n-1,n-1,1,n-1} = 1$. Also, in this row of R, we see that $R_{n,n-1} = n-2$. Therefore, we let $a_{n-1,n-1,n-1,n-2} = 1$.

Thus, we set equal to 1 the entries of $\mathcal{A}$ with indices

$$(i,i,0,0),(i,i,1,1),(i,i,2,2),\ldots,(i,i,n-1,n-1), \qquad i=0,\ldots,n-3$$

and

$$(i,i,0,0),(i,i,1,n-1),(i,i,2,1),(i,i,3,2),\ldots,(i,i,n-1,n-2), \qquad i=n-2,n-1.$$

We choose these entries as a positive diagonal for B_i, where $i=0,\ldots,n-1$, and denote them by 1_{s2} in Table 1. So far, the entries of $\mathcal{A}$ with the following indices are equal to 1. (We set these entries equal to 1 in steps 1 and 2.)

$$\begin{array}{ccccc}
(0,0,0,0) & (0,0,1,1) & (0,0,2,2) & \ldots & (0,0,n-1,n-1), \\
(1,1,0,0) & (1,1,1,1) & (1,1,2,2) & \ldots & (1,1,n-1,n-1), \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
(n-3,n-3,0,0) & (n-3,n-3,1,1) & (n-3,n-3,2,2) & \ldots & (n-3,n-3,n-1,n-1), \\
(n-2,n-2,0,0) & (n-2,n-2,1,n-1) & (n-2,n-2,2,1) & \ldots & (n-2,n-2,n-1,n-2), \\
(n-1,n-1,0,0) & (n-1,n-1,1,n-1) & (n-1,n-1,2,1) & \ldots & (n-1,n-1,n-1,n-2),
\end{array}$$

Also, we assume that for all other indices of the form

$$(i,i,\lambda,\mu), \qquad i=0,1,\ldots,n-1, \quad \lambda,\mu=1,\ldots,n-1,$$

the entries of $\mathcal{A}$ are equal to 0. We denote these entries by 0_{s2} in Table 1. We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A})=0$.

Step 3: We let $a_{*,n-2,k,k}=a_{*,n-1,k,k}=0$, where $*=0,\ldots,n-3$ and $k=0,\ldots,n-1$, since these entries are on the same horizontal line as $a_{*,*,k,k}=1$, where $*=0,\ldots,n-3$. We denote these 0 entries by 0_{s3} in Table 1.

Step 4: Consider the 2-dimensional plane C_k of the form

$$C_k=\mathcal{A}(*_1,*_2,k,k), \qquad k=1,\ldots,n-1, \qquad *_1,*_2=0,1,\ldots,n-1.$$

As before, since $\mathcal{A}$ is a polystochastic tensor, C_k is a doubly stochastic matrix. We set equal to 1 the entries of $\mathcal{A}$ with the following indices in steps 1 and 2.

$$(0,0,k,k),(1,1,k,k),(2,2,k,k),\ldots,(n-3,n-3,k,k), \qquad k=1,\ldots,n-1. \tag{2}$$

The entries of $\mathcal{A}$ with the indices mentioned in (2) form a positive partial diagonal of length $n-2$ for C_k, where $k=1,\ldots,n-1$. We can extend each of these positive partial diagonals of length $n-2$ to a positive partial diagonal of length $n-1$ by Lemma 3.9. Thus, we let $a_{n-2,n-1,k,k}=1$ for $k=1,\ldots,n-1$, and extend each of these positive partial diagonals of length $n-2$ to a positive partial diagonal of length $n-1$ in the form

$$(0,0,k,k),(1,1,k,k),(2,2,k,k),\ldots,(n-3,n-3,k,k),(n-2,n-1,k,k), \quad k=1,\ldots,n-1.$$

We denote these entries by 1_{s4} in Table 1. At the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A})=0$.

Step 5: Now, we consider the vertical lines of $\mathcal{A}$ with indices $(*,n-2,k,k)$, where $k=1,\ldots,n-1, *=0,\ldots,n-1$. We set equal to 0 the entries with indices $(*,n-2,k,k)$, where $*=0\ldots,n-2$ and $k=1,\ldots,n-1$, in steps 2 and 3. In order for $\mathcal{A}$ to remain polystochastic, we have to let $a_{n-1,n-2,k,k}=1$

for $k = 1, \ldots, n-1$. We denote these entries by 1_{s5} in Table 1. But, this gives a positive diagonal for $\mathcal{A}$ of the form

$$a_{0,0,0,0}, a_{1,1,1,1}, \ldots, a_{n-3,n-3,n-3,n-3}, a_{n-2,n-1,n-2,n-2}, a_{n-1,n-2,n-1,n-1}.$$

Therefore, we cannot construct a 4-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0 by using special row-Latin rectangle R. The reason is that if $a_{n-1,n-2,n-1,n-1} = 0$, then $\mathcal{A}$ cannot be polystochastic. Thus, the permanent of each 4-dimensional polystochastic $(0,1)$-tensor of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive. $\square$

In the proof of Theorem 3.11, we can use any $n \times (n-1)$ row-Latin rectangle R' with no transversals and not necessarily isotopic to R. In this case, we can study Theorem 3.11 in the following form. Similar to the proof of Theorem 3.11, we first choose a positive diagonal for each doubly stochastic submatrix of $\mathcal{A}$ by using R' to set equal to 1 some entries of $\mathcal{A}$ that do not form a positive diagonal for it. Then, in order for $\mathcal{A}$ to be a polystochastic tensor, we consider each line of $\mathcal{A}$ whose all entries are equal to 0, and we set equal to 1 one entry of each of these lines in such a way that the positive entries considered so far do not form a positive diagonal for $\mathcal{A}$. Eventually, after these selections, we may obtain a positive diagonal for $\mathcal{A}$.

In Table 1, the table associated to Theorem 3.11, "1" denotes an entry equal to 1, "0" shows an entry equal to 0, and dots are used to denote insignificant entries. The indices determine the steps in which the entries are specified.

Table 1: The table associated to Theorem 3.11

$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0_{s3} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 0_{s3} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & 0_{s3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \iddots & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \iddots & \cdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & 0_{s3} & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & 0_{s3} & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0_{s3} & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \end{smallmatrix}$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0_{s3} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 0_{s3} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & 0_{s3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \iddots & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \iddots & \cdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & 0_{s3} & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & 0_{s3} & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0_{s3} & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \end{smallmatrix}$	$\cdots$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0_{s3} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 0_{s3} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & 0_{s3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \iddots & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \iddots & \cdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & 0_{s3} & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & 0_{s3} & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0_{s3} & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \end{smallmatrix}$	$\begin{smallmatrix} \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 1_{s4} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & 1_{s4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & 1_{s4} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \iddots & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \iddots & \cdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & 1_{s4} & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & 1_{s4} & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0_{s3} & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \end{smallmatrix}$	$\begin{smallmatrix} \cdot & 1_{s2} & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} \\ \cdot & 0_{s2} & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 1_{s2} \\ \cdot & 0_{s2} & 0_{s2} & 0_{s2} & \cdots & 0_{s2} & 1_{s2} & 0_{s2} \\ \cdot & 0_{s2} & 0_{s2} & 0_{s2} & \iddots & 1_{s2} & 0_{s2} & 0_{s2} \\ \vdots & \vdots & \vdots & \iddots & \iddots & \vdots & \vdots & \vdots \\ \cdot & 0_{s2} & 0_{s2} & 1_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} \\ \cdot & 0_{s2} & 1_{s2} & 0_{s2} & \cdots & 0_{s2} & 0_{s2} & 0_{s2} \\ 1_{s1} & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \end{smallmatrix}$
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The conjecture below was proposed by Wanless [16] in 2011.

Conjecture 3.12. Every odd-dimensional Latin hypercube of order n has a transversal.

The following conjecture is equivalent to Wanless' conjecture.

Conjecture 3.13. The permanent of an even-dimensional polystochastic $(0,1)$-tensor of order n is positive.

In what follows, we use the special row-Latin rectangle R to present an algorithm for the study of Conjecture 3.13.

Theorem 3.14. *The permanent of an even-dimensional polystochastic* $(0,1)$*-tensor of order* n *that is constructed using the special* $n \times (n-1)$ *row-Latin rectangle* R*, mentioned in Example 3.7, is positive.*

Proof. We show that the permanent of each $2s$-dimensional polystochastic $(0,1)$-tensor of order n, where $s \in \mathbb{N}$, is positive. We prove this theorem by induction. Since 2-dimensional polystochastic $(0,1)$-tensors are doubly stochastic matrices, the statement of the theorem is true for 2-dimensional polystochastic $(0,1)$-tensors of order n (that is, $L = 1$) by [9].
The induction hypothesis: Assume that the statement of the theorem is true for $2s-2$-dimensional polystochastic $(0,1)$-tensors of order n (that is, $L = s-1$). Thus, the permanent of each $2s-2$-dimensional polystochastic $(0,1)$-tensor, where $s \in \mathbb{N}$, is positive.

We must prove that the statement of the theorem is also true for $2s$-dimensional polystochastic $(0,1)$-tensors of order n (that is, $L = s$). To see this, we show that a $2s$-dimensional polystochastic $(0,1)$-tensor of order n, where $s \in \mathbb{N}$, with permanent equal to 0 cannot be constructed. In fact, we try to construct such a tensor, and we observe that this construction is not possible.

We follow the algorithm below step by step to construct a $2s$-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0. This algorithm has five steps. In step 1, we define a doubly stochastic matrix B, and we select a positive diagonal for B. In step 2, we define the $2s-2$-dimensional polystochastic tensors B_i and we select a positive diagonal for B_i, where $i = 0, \dots, n-1$, using a row-Latin rectangle R with no transversals. In step 4, we define the doubly stochastic matrices C_k, where $k = 1, \dots, n-1$, and we extend a positive partial diagonal of length $n-2$ to a positive partial diagonal of length $n-1$ for C_k, where $k = 1, \dots, n-1$. In step 5, we consider the vertical lines $(*, n-2, k, \dots, k)$, where $k = 1, \dots, n-1$ and $* = 0, \dots, n-1$, and we set equal to 1 the entries on these lines in order for $\mathcal{A}$ to remain polystochastic. In each step, we set some entries of $\mathcal{A}$ equal to 1, since $\mathcal{A}$ is a $(0,1)$-tensor by our hypothesis. At the end of step 5, we observe that we cannot construct a $2s$-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0.

Step 1: Consider the 2-dimensional plane B of the form

$$B = \mathcal{A}(*_1, *_2, \underbrace{0, \dots, 0}_{2s-2}) \qquad *_1, *_2 = 0, \dots, n-1.$$

Since $\mathcal{A}$ is a polystochastic tensor, B is a doubly stochastic matrix. Hence, by [9], B has a positive diagonal. Without loss of generality, assume that the entries of $\mathcal{A}$ with indices $(i, i, 0, 0, \dots, 0)$ are equal to 1, where $i = 0, \dots, n-1$, and form a positive diagonal for B.

We check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step 2: Consider the $2s-2$-dimensional plane $\mathcal{B}_i$ of the form

$$\mathcal{B}_i = \mathcal{A}(i, i, *_1, \dots, *_{2s-2}), \qquad i = 0, \dots, n-1, \qquad *_1, \dots, *_{2s-2} = 0, \dots, n-1.$$

Then, $\mathcal{B}_i$ is a $2s-2$-dimensional tensor of order n. Since $\mathcal{A}$ is a polystochastic $(0,1)$-tensor, $\mathcal{B}_i$ is a $2s-2$-dimensional polystochastic $(0,1)$-tensor of order n. The induction hypothesis allows us to conclude that the permanent of $\mathcal{B}_i$ is positive, where $i = 0, \dots, n-1$. To select a positive diagonal for $\mathcal{B}_i$, we consider an $n \times (n-1)$ row-Latin rectangle R with no transversals. (Since a transversal in R gives a positive diagonal for $\mathcal{A}$, and our goal is to construct a polystochastic $(0,1)$-tensor with permanent equal to 0, we consider a row-Latin rectangle without any transversals.) We assume that the entries of $\mathcal{A}$ with indices $\{(i, i, \beta_i^j, \gamma_i^j, \beta_i^j, \gamma_i^j, \dots, \beta_i^j, \gamma_i^j)\}_{j=1}^n$ are equal to 1, and also form a positive diagonal for $\mathcal{B}_i$ containing $a_{i,i,0,0,\dots,0}$, where β_i^j and γ_i^j are determined according to the rectangle R as follows.

The entry in the $(i+1)$th row and the β_i^jth column of R is γ_i^j, where $i = 0, \dots, n-1$.

In what follows, we introduce a row-Latin rectangle R, and we use it to choose β_i^j and γ_i^j.

We choose the row-Latin rectangle R mentioned in Example 3.7. (We know that R has no transversals.) We consider a positive diagonal for $\mathcal{B}_i$, where $i = 0, \dots, n-1$, according to the row-Latin rectangle R and the discussion above, as follows.

For example, to select a positive diagonal for $\mathcal{B}_0$, we look at the first row of R. In this row, we see that the entry located in the first column is equal to 1 (that is, $R_{11} = 1$). Hence, we let $a_{0011\dots1} = 1$. Also in this row of R, we see that $R_{1,n-1} = n-1$. Therefore, we let $a_{0,0,n-1,n-1,\dots,n-1} = 1$.

For instance, to choose a positive diagonal for $\mathcal{B}_{n-1}$, we look at the last row of R. In this row, we see that the entry located in the first column is equal to $n-1$ (that is, $R_{n1} = n-1$). So, we let $a_{n-1,n-1,1,n-1,1,n-1,\dots,1,n-1} = 1$. Also in this row of R, we see that $R_{n,n-1} = n-2$. Therefore, we let

$$a_{n-1,n-1,n-1,n-2,n-1,n-2,\dots,n-1,n-2} = 1.$$

Thus, we set equal to 1 the entries of $\mathcal{A}$ with indices

$$(i,i,0,0,\dots,0), (i,i,1,1,\dots,1), (i,i,2,2,\dots,2), \dots, (i,i,n-1,n-1,\dots,n-1),$$

where $i = 0, \dots, n-3$, and

$$\begin{aligned}&(i,i,0,\dots,0), (i,i,1,n-1,1,n-1,\dots,1,n-1), (i,i,2,1,2,1,\dots,2,1),\\ &(i,i,3,2,3,2,\dots,3,2), \dots, (i,i,n-1,n-2,n-1,n-2,\dots,n-1,n-2), \; i = n-2, n-1,\end{aligned}$$

and we choose these entries as a positive diagonal for B_i, where $i = 0, \dots, n-1$.

So far, the entries of $\mathcal{A}$ with the following indices are equal to 1. (We set these entries equal to 1 in steps 1 and 2.)

$$\begin{array}{lllll}
(0,0,\dots,0) & (0,0,1,\dots,1) & (0,0,2,\dots,2) & \dots & (0,0,n-1,\dots,n-1),\\
(1,1,0,\dots,0) & (1,1,\dots,1) & (1,1,2,\dots,2) & \dots & (1,1,n-1,\dots,n-1),\\
\vdots & \vdots & \vdots & \dots & \vdots\\
(n-3,n-3,0,\dots,0) & (n-3,n-3,1,\dots,1) & (n-3,n-3,2,\dots,2) & \dots & (n-3,n-3,n-1,\dots,n-1),\\
(n-2,n-2,0,\dots,0) & (n-2,n-2,1,n-1,1,n-1,\dots,1,n-1) & (n-2,n-2,2,1,2,1,\dots,2,1) & \dots & (n-2,n-2,n-1,n-2,n-1,n-2,\dots,n-1,n-2),\\
(n-1,n-1,0,\dots,0) & (n-1,n-1,1,n-1,1,n-1,\dots,1,n-1) & (n-1,n-1,2,1,2,1,\dots,2,1) & \dots & (n-1,n-1,n-1,n-2,n-1,n-2,\dots,n-1,n-2).
\end{array}$$

Also, we assume that for all other indices of the form

$$(i, i, \lambda_1, \lambda_2, \dots, \lambda_{2s-2}), \qquad i = 0, 1, \dots, n-1, \quad \lambda_1, \lambda_2 \dots, \lambda_{2s-2} = 1, \dots, n-1,$$

the entries of $\mathcal{A}$ are equal to 0. Notice that at the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step 3: We let $a_{*,n-2,k,\dots,k} = a_{*,n-1,k,\dots,k} = 0$, where $* = 0,\dots,n-3$ and $k = 0,\dots,n-1$, since these entries are on the same horizontal line as $a_{*,*,k,k,\dots,k} = 1$, where $* = 0,\dots,n-3$.

Step 4: Consider the 2-dimensional plane C_k of the form

$$C_k = \mathcal{A}(*_1, *_2, \underbrace{k,\dots,k}_{2s-2}), \qquad k = 1,\dots,n-1, \qquad *_1, *_2 = 0,1,\dots,n-1.$$

As before, C_k is a doubly stochastic matrix. We set equal to 1 the entries of $\mathcal{A}$ with the following indices in steps 1 and 2.

$$\begin{aligned}&(0,0,k,k,\dots,k),(1,1,k,k,\dots,k),(2,2,k,k,\dots,k),\dots,\\&(n-3,n-3,k,k,\dots,k),\quad k=1,\dots,n-1.\end{aligned} \tag{3}$$

So, the entries of $\mathcal{A}$ with indices mentioned in (3) form a positive partial diagonal of length $n-2$ for C_k, where $k = 1,\dots,n-1$. We can extend each of these positive partial diagonals of length $n-2$ to a positive partial diagonal of length $n-1$ by Lemma 3.9. Hence, we let $a_{n-2,n-1,k,k,\dots,k} = 1$, where $k = 1,\dots,n-1$, and extend each of these positive partial diagonals of length $n-2$ to a positive partial diagonal of length $n-1$ of the form

$$\begin{aligned}&(0,0,k,k,\dots,k),(1,1,k,k,\dots,k),(2,2,k,k,\dots,k),\dots,\\&(n-3,n-3,k,k,\dots,k),(n-2,n-1,k,k,\dots,k),\qquad k=1,\dots,n-1.\end{aligned}$$

At the end of this step, we check the permanent of $\mathcal{A}$ and observe that $per(\mathcal{A}) = 0$.

Step 5: Now, we consider the vertical lines of $\mathcal{A}$ with indices $(*, n-2, k, k,\dots,k)$, where $k = 1,\dots,n-1$, $* = 0,\dots,n-1$. We set equal to 0 the entries with indices $(*, n-2, k, k,\dots,k)$, where $* = 0,\dots,n-2$ and $k = 1,\dots,n-1$, in steps 2 and 3. In order for $\mathcal{A}$ to remain polystochastic, we have to let $a_{n-1,n-2,k,k,\dots,k} = 1$ for $k = 1,\dots,n-1$. But, this gives a positive diagonal for $\mathcal{A}$ with the following indices.

$$\begin{aligned}(0,0\dots,0),(1,1\dots,1),\dots,(n-3,n-3,\dots,n-3),&(n-2,n-1,n-2,n-2\dots,n-2),\\&(n-1,n-2,n-1,n-1\dots,n-1).\end{aligned}$$

So, we cannot construct a $2s$-dimensional polystochastic $(0,1)$-tensor of order n with permanent equal to 0. Thus, the permanent of each even-dimensional polystochastic $(0,1)$-tensor of order n that is constructed using the special $n \times (n-1)$ row-Latin rectangle R, mentioned in Example 3.7, is positive. □

In the proof of Theorem 3.14, we can use any $n \times (n-1)$ row-Latin rectangle R' with no transversals and not necessarily isotopic to R. In this case, we can study Theorem 3.14 in the following form. Similar to the proof of Theorem 3.14, we first choose a positive diagonal for each 2-dimensional plane and $2s-2$-dimensional plane of $\mathcal{A}$ by using R' to set equal to 1 some entries of $\mathcal{A}$ that do not form a positive diagonal for it. Then, in order for $\mathcal{A}$ to be a polystochastic tensor, we consider each line of $\mathcal{A}$ whose all entries are equal to 0, and we set equal to 1 one entry of each of these lines in such a way that the positive entries considered so far do not form a positive diagonal for $\mathcal{A}$. Eventually, after these selections, we may obtain a positive diagonal for $\mathcal{A}$.

According to the relation between $d+1$-dimensional polystochastic $(0,1)$-tensors of order n and d-dimensional Latin hypercubes of the same order, we obtain the following corollary (special case of Wanless' conjecture).

Corollary 3.15. *The number of transversals in an odd-dimensional Latin hypercube of order* 4 *is positive.*

Proof. The number of transversals in an odd-dimensional Latin hypercube of order n is equal to the permanent of its corresponding even-dimensional polystochastic $(0,1)$-tensor of order n. We know that the row-Latin rectangle R is the only $n \times (n-1)$ row-Latin rectangle with no transversals for $n = 4$ [14]. Also, By Theorem 3.14, we know that the permanent of each even-dimensional polystochastic $(0,1)$-tensor of order n is positive. Thus, the number of transversals in an odd-dimensional Latin hypercube of order 4 is positive. □

Also, Taranenko proved Corollary 3.15 by using of Quasigroups [15].

4 Hypergraphs

As is known, for $d \geq 3$, it is an NP-complete problem to determine whether a given bipartite hypergraph contains a perfect matching. In this section, we discuss the application of the permanents of tensors to hypergraphs, and we give a positive answer to this question for the bipartite hypergraph associated to an even-dimensional polystochastic $(0,1)$-tensor of order 4.

Definition 4.1 ([13])**.** A pair $H = (X, W)$ is called a *hypergraph* with *vertex set* X and *hyperedge set* W, where each hyperedge $w \in W$ is a subset of the vertices in X. A hypergraph H is called k*-uniform* if each of its hyperedges consists of k vertices.

Definition 4.2 ([4])**.** Let $\mathcal{A}$ be a d-dimensional tensor of order n. The generalization of the bipartite graph of a matrix to the tensor is the hypergraph $H(\mathcal{A}) = (V, W)$ with the vertex set

$$V = \{v_j^k : k = 1, \dots, d, j = 1, 2, \dots, n_k\},$$

and the edge set

$$W = \{(v_{i_1}^1, v_{i_2}^2, \dots, v_{i_d}^d) : a_{i_1 i_2 \dots i_d} \neq 0)\}.$$

Proposition 4.3 ([4])**.** The bipartite hypergraph introduced above is a simple hypergraph.

Definition 4.4 ([4])**.** For a simple hypergraph $H = (V, W)$, a subset M of the edges of H is said to be a *matching* if the edges in M are pairwise disjoint. A matching M is said to be *perfect* if M is a partition of the vertices of H. Observe that if $\mathcal{A}$ is a square d-dimensional tensor of order n, then the perfect matchings of $H(\mathcal{A})$ are the matchings of $H(\mathcal{A})$ of cardinality n. Let $\mathcal{A}$ be a d-dimensional tensor of order n. A non-zero term in the expansion (1) of the permanent of $\mathcal{A}$ corresponds to a matching $\{\{v_i^1, v_{\sigma_2(i)}^2, \dots, v_{\sigma_d(i)}^d\} : i = 1, \dots, n\}$ of the hypergraph $H(\mathcal{A})$ of cardinality n. Indeed, $per(\mathcal{A})$ is the sum of all products of n entries of $\mathcal{A}$ corresponding to the edges in the matching of $H(\mathcal{A})$ of cardinality n. If $\mathcal{A}$ is a d-dimensional $(0,1)$-tensor of order n, then $per(\mathcal{A})$ is the number of the matchings of $H(\mathcal{A})$ of cardinality n. Clearly, if $\mathcal{A}$ is a d-dimensional $(0,1)$-tensor of order n, then $per(\mathcal{A})$ is the number of the perfect matchings of $H(\mathcal{A})$ of cardinality n.

We know that the row-Latin rectangle R is the only $n \times (n-1)$ row-Latin rectangle with no transversals for $n = 4$ [14]. Thus, using Theorem 3.14 we obtain the following corollary.

Corollary 4.5. *The number of perfect matchings of the bipartite hypergraph associated to an even-dimensional polystochastic* $(0,1)$*-tensor of order* 4 *is positive.*

5 Conclusion

In this paper, we proved the positivity of the permanent of a 4-dimensional polystochastic $(0,1)$-tensor of order n that was constructed using a special $n \times (n-1)$ row-Latin rectangle R with no transversals. Also, we established the positivity of the permanent of an even-dimensional polystochastic $(0,1)$-tensor of order n that was constructed using the row-Latin rectangle R. Although we proved these theorems by using the special $n \times (n-1)$ row-Latin rectangle R with no transversals, we presented an algorithm to study these theorems by using each $n \times (n-1)$ row-Latin rectangle R' with no transversals and not necessarily isotopic to R. In the special case $n = 4$, we proved the positivity of the permanent of an even-dimensional polystochastic $(0,1)$-tensor of order 4 (answering Wanless' conjecture for odd-dimensional Latin hypercubes of order 4). Moreover, we applied our main theorem to the theory of hypergraphs; we proved the positivity of the number of perfect matchings of the bipartite hypergraph associated to an even-dimensional polystochastic $(0,1)$-tensor of order 4.

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