Matching Energy of Graphs with Maximum Degree at Most 3

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Abstract

The matching energy of a graph G, denoted by $ME(G)$, is defined as the sum of absolute values of the zeros of the matching polynomial of G. In this paper, we prove that if G is a connected graph of order n with maximum degree at most 3, then $ME(G) > n$ with only six exceptions.

Keywords: Matching energy, Matching polynomial

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1 Introduction

All graphs we consider are finite, simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. By *order* and *size* of G, we mean the number of vertices and the number of edges of G, respectively. We denote the complete graph, the path and the cycle of order n, by K_n, P_n and C_n, respectively. A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. For vertex disjoint graphs H and K, we use $H \cup K$ to denote their union. By mG we mean the graph consisting of m pairwise disjoint copies of G. The maximum degree of G is denoted by $\Delta(G)$ (or by Δ if G is clear from the context). For $S \subseteq V(G)$, $\langle S \rangle$ is the subgraph of G induced by S. A *traceable graph*, is a graph with a Hamilton path. A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. An *r-matching* in a graph G is a set of r pairwise non-incident edges. The number of r-matchings in G is denoted by $m(G, r)$. The *matching number* of G, $\mu(G)$, is the number of edges in a maximum matching of G.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a graph G, i.e the eigenvalues of its adjacency matrix. The energy of the graph G denoted by $\mathcal{E}(G)$, is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The theory of graph energy is well developed nowadays, for details see [2, 3, 9, 11, 17]. The Coulson integral formula [8] plays an important role in the study on graph energy, its version for an acyclic graph T is as follows:

[1] speaker

$$\mathcal{E}(T) = \frac{2}{\pi}\int_0^{+\infty} \frac{1}{x^2} \ln\left(\sum_{r\geq 0} m(T,r)x^{2r}\right) dx. \tag{1}$$

Motivated by formula (1), Gutman and Wagner in 2012 defined the matching energy of a graph G as

$$ME(G) = \frac{2}{\pi}\int_0^{+\infty} \frac{1}{x^2} \ln\left(\sum_{r\geq 0} m(G,r)x^{2r}\right) dx, \tag{2}$$

see [12]. Energy and matching energy of graphs are closely related, and they are two quantities of relevance for chemical applications, [12]. Recall that the *matching polynomial* of G is defined by

$$\alpha(G,x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r m(G,r)x^{n-2r},$$

where n is the order of G and $m(G,0)$ is considered to be 1, see [1, 4, 5, 6, 7, 10]. For any graph G, all zeros of $\alpha(G,x)$ are real [14]. Furthermore, if μ is a matching zero of G, then so is $-\mu$. The following result gives an equivalent definition of matching energy:

Theorem 1.1. *[12] Let G be a graph and let $\mu_1,\ldots,\mu_n$ be the zeros of its matching polynomial. Then*

$$ME(G) = \sum_{i=1}^{n} |\mu_i|.$$

Since 2012 matching energy of graphs has been studied by several authors and a series of results concerning the extremal matching energy of graphs have been obtained. For details, we refer to [12, 15, 16, 18]. Recently, in [13] the authors presented some lower bounds for matching energy of graphs. They proved that for a connected graph G, $ME(G) \geq 2\mu(G)$. Also it was shown that if G has no perfect matching, then $ME(G) \geq 2\mu(G) + 1$, except for $K_{1,2}$. Furthermore, a lower bound for $ME(G)$ in terms of the minimum degree of G was given. Among other results, they characterized some class of graphs whose matching energy exceeds the number of vertices. They proved that if G is a connected graph of order n such that the multiplicity of 0 as a matching root is 1, then $ME(G) > n$, except for $K_{1,2}$. Also it was shown that in a connected graph G, if the multiplicity of 0 as a matching root is 2, then except four graphs, the matching energy of G exceeds the number of vertices. In particular, all connected traceable graphs and all connected claw-free graphs whose matching energies are greater than the number of vertices were described. In this paper, we characterize all connected graphs with maximum degree at most 3, whose matching energies are equal to the number of vertices. In fact, we show that if G is a connected graph of order n with maximum degree at most 3, then $ME(G) > n$ with only six exceptions. The following lemmas are needed in the sequel.

Lemma 1.2. *[18] If H is a subgraph of G, then $ME(H) \leq ME(G)$, with equality if H and G are the same except possibly for isolated vertices.*

Lemma 1.3. *Let G be a connected graph and $H_1,\ldots,H_t$ be its t vertex-disjoint subgraphs. Then*

$$ME(G) > \sum_{i=1}^{t} ME(H_i).$$

Proof. Let $K = \cup_{i=1}^{t} H_i$. Now the assertion follows from Lemma 1.2 and [13, Lemma 16]. □

Lemma 1.4. *[13] Let G be a connected graph of order n witch has a perfect matching. Then $ME(G) \geq n$ and the equality holds only if $G = K_2$.*

Lemma 1.5. *[13] Let G be a connected traceable graph of order $n > 1$. Then $ME(G) \geq n$, except for $K_{1,2}$. The equality holds only if $G = K_2$.*

Lemma 1.6. *[13] Let $n \geq 3$. Then $ME(C_n) > n$. In particular, if n is even, then $ME(C_n) > n + 1$.*

2 Main results

In this section, we characterize all connected graphs with maximum degree at most 3, whose matching energies are equal to the number of vertices. It is shown that if G is a connected graph of order n with maximum degree at most 3, then $ME(G) > n$, with only six exceptions.

Lemma 2.1. *Let G be a connected graph of order n with $\Delta \leq 2$. Then $ME(G) > n$, except for K_1, K_2 and $K_{1,2}$. The equality holds only if $G = K_2$.*

Proof. Since $\Delta \leq 2$, G is either a path or a cycle. Now, the assertion follows from Lemmas 1.5 and 1.6. □

Lemma 2.2. *Let G be a connected graph of order $n > 2$. If G has a perfect matching, then $ME(G) \geq n + 0.47$.*

Proof. Let M be a perfect matching of G and $e = uv$ be a P_2-component of M. Since G is connected, there exists some P_2-component of M, say $f = wz$ such that e is connected to f. Let $H = \langle u, v, w, z\rangle$. A computer search shows that $ME(H) \geq 4.47$.

Now, if $G = H$, then we are done. Otherwise, let $W = G \setminus V(H)$ and assume that $W_1, \ldots, W_t$, $t \geq 1$ are the components of W. Now, Lemma 1.4, implies that for each i, $1 \leq i \leq t$, $ME(W_i) \geq |V(W_i)|$. Consequently, by Lemma 1.3 we obtain:

$$ME(G) > ME(H) + \sum_{i=1}^{t} ME(W_i) \geq (|V(H)| + 0.47) + \sum_{i=1}^{t} |V(W_i)| = n + 0.47,$$

so we are done. □

Lemma 2.3. *Let G be a connected graph with $\Delta = 3$. Then there are vertex disjoint subgraphs $H_1, \ldots, H_l$ of G such that $V(G) = \cup_{i=1}^{l} V(H_i)$ and each H_i is isomorphic to one of the graphs K_2, $K_{1,2}$, C_3 or $K_{1,3}$.*

Proof. Let M be a matching of maximum size in G and $|M| = l$. Let $e_1, \ldots, e_l$ be the P_2-components of M. Assume that S is the set of vertices of G missed by M and $|S| = k$. Obviously, S is an independent set of G. If $k = 0$, then G has a perfect matching and we are done. Hence we may assume that $k > 0$. Since G is connected each $x \in S$ is connected to some P_2-component of M.

Let S_1 be the set of vertices of S which are connected to e_1. For each $i > 1$, let S_i be the set of vertices of $S \setminus \cup_{j=1}^{i-1} S_j$ which are connected to e_i. Since $\Delta = 3$, it is easily seen that $|S_i| \leq 2$. Otherwise G has a matching of size at least $l + 1$, a contradiction. Now, let $H_i = \langle S_i, V(e_i)\rangle$, for $i = 1, \ldots, l$. This implies the statement. □

Now, we are ready to state the next theorem.

Theorem 2.4. *Let G be a connected graph of order $n \geq 4$ with $\Delta = 3$. If $G \notin \{K_{1,3}, G_1, G_2\}$, then $ME(G) > n$. In particular, G_1 is the only connected graph with $\Delta = 3$ whose matching energy is equal to its order.*

G_1 G_2

References

[1] S. Akbari, P. Csikvári, A. Ghafari, S. Khalashi Ghezelahmad, M. Nahvi, Graphs with integer matching polynomial zeros, Discrete Appl. Math. 224 (2017) 1-8.

[2] S. Akbari, S. Khalashi Ghezelahmad, Non-Hypoenergetic Graphs with Nullity 2, MATCH Commun. Math. Comput. Chem. 87 (2022) 717-727.

[3] S. Akbari, M. Ghahremani, M.A. Hosseinzadeh, S. Khalashi Ghezelahmad, H. Rasouli, A. Tehranian, A lower bound for graph energy in terms of minimum and maximum degrees, MATCH Commun. Math. Comput. Chem. 86 (2021) 549-558.

[4] E.J. Farrell, An introduction to matching polynomials, J. Comb. Theory B 27 (1979) 75-86.

[5] C.D. Godsil, Algebraic Combinatorics, Vol. 6, CRC Press, 1993.

[6] C.D. Godsil, Algebraic matching theory, Electron. J. Combin. 2 (1995) 1-14.

[7] C.D. Godsil, I. Gutman, On the theory of the matching polynomial, J. Graph Theory 5(2) (1981) 137-144.

[8] I. Gutman, Acyclic systems with extremal Hückel -electron energy, Theor. Chim. Acta. 45 (1977) 79-87.

[9] I. Gutman, The energy of a graph, Ber. Math. Statist. Sekt. Forschungsz. Graz. 103 (1978) 1-22.

[10] I. Gutman., The matching polynomial, MATCH Commun. Math. Comput. Chem. 6 (1979) 75-91.

[11] I. Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, 196-211.

[12] I. Gutman, S. Wagner, The matching energy of a graph, Discrete Appl. Math. 160 (2012) 2177-2187.

[13] S. Khalashi Ghezelahmad, Lower bounds on matching energy of graphs, Discrete Appl. Math. 307 (2022) 153-159.

[14] O.J. Heilmann, E.H. Lieb, Theory of monomer-dimer systems, Commun. Math. Physics, 25 (1972) 190-232.

[15] H. Li, Y. Zhou, Li. Su, Graphs with extremal matching energies and prescribed parameters, MATCH Commun. Math. Comput. Chem. 72 (2014) 239-248.

[16] S. Li, W. Yan, The matching energy of graphs with given parameters, Discrete Appl. Math. 162 (2014) 415-420.

[17] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.

[18] W. Wang, S. Wasin, On minimum matching energy of graphs, MATCH Commun. Math. Comput. Chem. 74 (2015) 399-410.

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