# The nS(G)-Autonilpotency Of Groups

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**Abstract**

The various series, definitions such as nilpotency and solubility of groups and all kinds of automorphisms have been the idea of many researchers articles. In this paper, we first study autonilpotent group and their generalizations. Then we give a new definition for nS(G)-autonilpotency and discuss some properties of this concept.

**Keywords:** IA-group, IA-central subgroup, autonilpotent group, S(G)-autonilpotent groups, nS(G)-autonilpotent groups

**AMS Mathematical Subject Classification [2010]:** 20D45, 20D15

## 1 Introduction

Let G be a group. Let us denote by $G'$, Z(G), and Aut(G), respectively the commutator subgroup, the centre and the full automorphism group. Let $H \leqslant G$, then

$$C_{Aut(G)}(H) = \{\alpha \in Aut(G) \mid \alpha(h) = h, \ \forall \ h \in H\}.$$

Bachmuth [1] in 1965 defined an IA-automorphism of a group G as

$$IA(G) = \left\{\alpha \in Aut(G) \,\middle|\, g^{-1}\alpha(g) = [g,\alpha] \in G', \ \forall \ g \in G\right\}.$$

Hegarty [4] in 1994 introduced the absolute center

$$L(G) = \left\{g \in G \,\middle|\, g^{-1}\alpha(g) = 1, \ \forall \ \alpha \in Aut(G)\right\}.$$

On the similar lines, Ghumde and Ghate [3] in 2015 introduced the IA-central subgroup

$$S(G) = \left\{g \in G \mid g^{-1}\alpha(g) = 1, \ \alpha \in IA(G)\right\}.$$

For any group G, $L(G) \trianglelefteq S(G) \trianglelefteq Z(G)$.

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# 2 Main results

The concept of autonilpotent groups were introduced by Parvaneh and Moghaddam [7] in 2010. They defined the upper autocentral series of G in the following way:

$$\langle 1\rangle = L_0(G) \subseteq L_1(G) = L(G) \subseteq L_2(G) \subseteq \cdots \subseteq L_n(G) \subseteq \cdots$$

where

$$\frac{L_n(G)}{L_{n-1}(G)} = L\Big(\frac{G}{L_{n-1}(G)}\Big), \qquad for\ n \geq 2$$

and $L_n(G)$ is nth-absolute centre of G. Also, they called a group to be autonilpotent of class at most c if $L_c(G) = G$, for some positive integer c.

For each natural number i and n, we [2] defined

$$L_i^n(G) = \{g \in G \mid [g, \alpha_1^n, \alpha_2^n, \ldots, \alpha_i^n] = 1,\ \forall\ \alpha_1, \alpha_2, \ldots, \alpha_i \in Aut(G)\}.$$

Also, we called a group G to be n-autonilpotent group of class at most c if there exists some positive integer c such that $L_c^n(G) = G$.

Thereafter, we define the IA-central series of G in the following way:

$$\langle 1\rangle = S_0(G) \subseteq S_1(G) = S(G) \subseteq S_2(G) \subseteq \cdots \subseteq S_i(G) \subseteq \cdots$$

where

$$S_i(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, \ldots, \alpha_i] = 1,\ \forall\ \alpha_1, \alpha_2, \ldots, \alpha_i \in IA(G)\}, \quad i \geq 1.$$

A group G is said to be S(G)-autonilpotent(or IA-nilpotent) group of class at most c if $S_c(G) = G$, for some positive integer number c.

In this section, we generalize the concept of S(G)-autonilpotency and represent their properties.

## 2.1 Preliminary Results

**Definition 2.1.** For each positive integer i and n, we define

$$S_i^n(G) = \{g \in G \mid [g, \alpha_1^n, \alpha_2^n, \ldots, \alpha_i^n] = 1,\ \forall\ \alpha_1, \alpha_2, \ldots, \alpha_i \in IA(G)\}.$$

**Definition 2.2.** A group G is said to be nS(G)-autonilpotent group of class at most c if $S_c^n(G) = G$, for some positive integer c.

**Example 2.3.** For an abelian group G, we know that IA(G) is trival, so $S_i^n(G) = G$, for every positive integer i. Therefore, abelian groups are nS(G)-autonilpotent.

**Remark 2.4.** Clearly, for a group G and every positive integer i, $L_i^n(G) \leqslant S_i^n(G)$. Thus n-autonilpotent groups are nS(G)-autonilpotent groups, but the converse of this result is not true in general.
For example $\mathbb{Z}_3$ is nS(G)-autonilpotent, but is not n-autonilpotent.

**Proposition 2.5.** *Let $G$ be any group, then for each $g \in G$ we have*

$$g \in S_i^n(G) \iff [g, \alpha] \in S_i^{n-1}(G), \quad \forall\, \alpha \in IA(G).$$

*Proof.* Due to $S_i^n(G)$ definition and by inductive on i, the lemma is proved. □

The following corollary is immediate consequence of the above proposition.

**Corollary 2.6.** *For each $g \in G$, we have*

$$g \in S_i^n(G) \iff [S_i^{n-1}(G), IA(G)] = 1.$$

**Lemma 2.7.** *Let $G$ be a non-trivial nS(G)-autonilpotent group, then $S(G) \neq \langle 1 \rangle$.*

*Proof.* By the assumption there exist a positive integer i such that $S_i^n(G) = G$. We assume by way of contradiction that $S(G) = \langle 1 \rangle$, then according to $S_i^n(G)$ definition and by lemma **??**, $S_2^n(G) = \langle 1 \rangle$. thus, we have $S_i^n(G) = \langle 1 \rangle$, for every positive integer i, contrary to the assumption. Hence $S(G) \neq \langle 1 \rangle$. □

**Theorem 2.8.** *If the group $G = H_1 \times H_2$ is the direct product of its characteristic subgroups $H_1$ and $H_2$, then for all $i \geq 1$,*

$$S_i^n(H_1 \times H_2) = S_i^n(H_1) \times S_i^n(H_2).$$

*Proof.* The Theorem holds by induction on i. □

**Corollary 2.9.** *If $H_1$ and $H_2$ be two finite groups such that $(|H_1|, |H_2|) = 1$, then*

$$S_i^n(H_1 \times H_2) = S_i^n(H_1) \times S_i^n(H_2).$$

**Corollary 2.10.** *If $G = H_1 \times H_2$ is the direct product of its characteristic subgroups such that $H_1$ or $H_2$ is not nS(G)-autonilpotent, then so is not $G$.*

**Corollary 2.11.** *If $G_1$, $G_2$, …, $G_k$ are nS(G)-autonilpotent groups with coprime orders, then so is*

$$G_1 \times G_2 \times \cdots \times G_k.$$

## 2.2 When $S_i^n(G) \neq \langle 1 \rangle$?

Now, we study the conditions in which $S_i^n(G)$ is non-trivial. We saw that for abelian groups $S_i^n(G) = G$, therefore, in the following, we consider non-abelian groups.

**Theorem 2.12.** *Let $G$ be a group and $H \leqslant G$, then $H \leqslant S_i^n(G)$ if one of the following conditions holds:*

1) *$Aut(G) = C_{Aut(G)}(H)$.*

2) *$G$ be a finite group and $H$ be a characteristic subgroup of prime order $p$ such that $p$ be the smallest prime divisor of $|Aut(G)|$.*

3) *$Aut(G)$ be a perfect group and $H$ be a cyclic characteristic subgroup of $G$.*

*Proof.* Given that $L(G) \leqslant S(G) \leqslant S_i^n(G)$, the proof easily follow from [6] lemma 2.4(iv), corollary 3.5 and 3.7, respectively. □

**Theorem 2.13.** *Let G be a group, Aut(G) be a finite p-group and H be a finite characteristic subgroup of G such that* $p\big||H|$, *then* $H \cap S_i^n(G) \neq \langle 1 \rangle$.

*Proof.* Because H is a characteristic subgroup of G, then this equivalence relation yields a partition of H and each cell in the partition arising from an equivalence relation is equivalence class. According to lemma 2.5 [6], there is $1 \neq h_0 \in H$ element such that the equivalence class is of order 1. So we have $\alpha(h_0) = h_0$, for every $\alpha \in Aut(G)$. Thus $1 \neq h_0 \in S(G) \cap H$ and this completes the proof. □

**Corollary 2.14.** *If G be a finite group such that Aut(G) is a p-group, then* $S_i^n(G) \neq \langle 1 \rangle$.

**Theorem 2.15** (MacHale[5])**.** *Let G be a finite group such that Aut(G) is nilpotent. If G is not cyclic of odd order, then G contains a non-trivial element which is left fixed by every automorphism of G.*

**Corollary 2.16.** *Let G be a finite group such that Aut(G) is nilpotent, then* $S_i^n(G) \neq \langle 1 \rangle$.

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