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The Generalize Of G^{**}-Autonilpotent Groups

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Abstract

Let G be a group. We introduced a series on the subgroups generated by G and IA(G) and gave a definition for G^{**}-autonilpotency on this series [2]. In this paper, we generalize this concepts and then we study some properties of them and their relationships.

Keywords: IA-group, n-IA-commutator series, autonilpotent groups, G^{**}-autonilpotent groups, n-IA-nilpotent groups

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1 Introduction

Let G be a group. Let us denote by G' and Aut(G), respectively the commutator subgroup and the full automorphism group. Bachmuth [1] in 1965 defined an IA-automorphism of a group G as

$$IA(G) = \left\{\alpha \in Aut(G) \,\middle|\, g^{-1}\alpha(g) = [g,\alpha] \in G',\ \forall\, g \in G\right\}.$$

Hegarty [4] in 1994 introduced the autocommutator subgroup as follows:

$$K(G) = \langle [g,\alpha] \mid g \in G,\ \alpha \in Aut(G)\rangle.$$

On the similar lines, Ghumde and Ghate [3] in 2015 introduced the subgroup

$$G^{**} = \langle [g,\alpha] \mid g \in G,\ \alpha \in IA(G)\rangle.$$

For any group G, $G' = G^{**} \leqslant K(G)$.

First, we study the conditions in which G^{**} is equal to K(G). The following proposition clearly states these conditions.

Proposition 1.1. *For a group G, $G^{**} = K(G)$ if one of the following conditions holds*

1) G be a complete group, i.e. $G = G'$.

2) $[G : G'] = 2$, because then IA(G)=Aut(G).

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3) Aut(G)=Inn(G).

For any group G, $G^{**} = \langle 1 \rangle$ if and only if G be a trivial or abelian group. Also, G^{**} is an abelian group if and only if G be a metabelian group.

Lemma 1.2. *Let $G = H_1 \times H_2$, $H_1 \neq \langle 1 \rangle$ and $H_1 \cap G^{**} = \langle 1 \rangle$, then $H_1 \cong C_2$.*

Proof. Suppose by way of contradiction that $|H_1| > 2$, then H_1 has a nontrivial automorphism α and α can be extended to an automorphism β of G where $\beta(h_1) = \alpha(h_1)$, for all $h_1 \in H_1$ and $\beta(h_2) = h_2$, for all $h_2 \in H_2$. If $h_1 \in H_1$ be arbitrary, then

$$[h_1, \beta] = [h_1, \alpha] \in H_1 \cap G^{**} = \langle 1 \rangle.$$

Thus $\alpha(h_1) = h_1$ and contradicting the fact that α is nontrivial. □

The concept of autonilpotent groups were introduced by Parvaneh and Moghaddam [6] in 2010. They defined the autocommutator subgroup of weight m+1 in the following way:

$$\begin{aligned} K_m(G) &= [K_{m-1}(G), Aut(G)] \\ &= \langle [g, \alpha_1, \alpha_2, \ldots, \alpha_m] \mid g \in G,\ \alpha_1, \alpha_2, \ldots, \alpha_m \in Aut(G) \rangle, \end{aligned}$$

for all $m \geq 1$, and obtained a descending chain of autocommutator subgroups of G as follows:

$$\cdots \subseteq K_m(G) \subseteq \cdots \subseteq K_2(G) \subseteq K_1(G) = K(G) \subseteq K_0 = G.$$

Also, they called a group to be autonilpotent of class at most m if $K_m(G) = G$, for some positive integer m.

Mohebian and Hosseini [5] in their paper generalized these concepts and defined the n-autocommutator subgroup inductively as follows:

$$\begin{aligned} K_0^n(G) =& G, \\ K_1^n(G) =& K^n(G) = \langle [g, \alpha^n] \mid g \in G,\ \alpha \in Aut(G) \rangle, \\ and\ for\ n \geq 2:\ K_m^n(G) =& \langle [g, \alpha_1^n, \ldots, \alpha_m^n] \mid g \in G,\ \alpha_1, \ldots, \alpha_m \in Aut(G) \rangle, \end{aligned}$$

in which $K_m^n(G) \overset{ch}{\leqslant} G$. Also, they introduced the lower n-autocenteral series of G as

$$\cdots \subseteq K_m^n(G) \subseteq \cdots \subseteq K_2^n(G) \subseteq K_1^n(G) = K^n(G) \subseteq K_0^n(G) = G$$

and they called a group G is n-autonilpotent group if the lower n-autocenteral series ends in the identity subgroup after a finite number of steps. In particular, for $n = 1$, we have the property of autonilpotent groups.

We [2] defined the IA-commutator series or G^{**} series of G in the following way:

$$\cdots \subseteq G_m^{**} \subseteq \cdots \subseteq G_2^{**} \subseteq G_1^{**} = G^{**} = G' \subseteq G_0^{**} = G \tag{1}$$

where m is a positive integer and

$$\begin{aligned} G_m^{**} =& \langle [g, \alpha_1, \ldots, \alpha_m] \mid g \in G,\ \alpha_1, \ldots, \alpha_m \in IA(G) \rangle \\ =& [G_{m-1}^{**}, IA(G)]. \end{aligned}$$

$G^{**}_m \leqslant K(G)$ and if G be an abelian group, then $G^{**}_m = \langle 1 \rangle$, for every positive integer m. A group G is said to be G^{**}-autonilpotent group of class at most m if the series (1) ends in the identity subgroup after a finite number of steps. The autonilpotent groups are G^{**}-autonilpotent, but the converse is not true in general. For example, all of abelian groups are G^{**}-autonilpotent, but

$$G = \bigoplus_{i=1}^{l} \mathbb{Z}_{2^{m_i}}, \qquad l > 1,\ \ m_1 = m_2 \geq \cdots \geq m_l$$

is not autonilpotent, because $K_n(G) = G$.

On the similar lines, we generalize these concepts and then we study their properties.

Definition 1.3. For each positive integer m and n, we define

$$G^{n*}_m = \langle [g, \alpha_1^n, \ldots, \alpha_m^n] \mid g \in G,\ \alpha_1, \ldots, \alpha_m \in IA(G) \rangle.$$

Thus, we have n-IA-commutator series as

$$\cdots \subseteq G^{n*}_m \subseteq \cdots \subseteq G^{n*}_2 \subseteq G^{n*}_1 = G^{n*} \subseteq G^{n*}_0 = G. \tag{2}$$

2 Properties of the terms of n-IA-commutator series

Proposition 2.1. *Let G be a group, then for every positive integer m and n, G^{n*}_m is a characteristic subgroup of G.*

Proof. Clearly, $G^{n*}_m \leqslant G$. Now, let $\beta \in Aut(G)$ and $[g, \alpha_1^n, \ldots, \alpha_m^n] \in G^{n*}_m$, for every $g \in G$ and $\alpha_1, \ldots, \alpha_n \in IA(G)$. Then, one can write

$$\begin{aligned}
\beta^n([g, \alpha_1^n, \ldots, \alpha_m^n]) &= \beta^n\big(g^{-1}\alpha_1^n \cdots \alpha_m^n(g)\big) \\
&= \beta^n(g^{-1})\beta^n\big(\alpha_1^n \cdots \alpha_m^n(g)\big) \\
&= \big(\beta(g)^{-n}\big)\beta^n\big(\alpha_1^n \cdots \alpha_m^n \beta^{-n}\beta^n(g)\big) \\
&= \big(\beta(g)^{-n}\big)\beta^n\alpha_1^n \cdots \alpha_m^n \beta^{-n}\big(\beta^n(g)\big) \\
&= [\underbrace{\beta(g)^{-n}}_{\in G}, \underbrace{\beta^n\alpha_1^n \cdots \alpha_m^n\beta^{-n}}_{\in IA(G) \trianglelefteq Aut(G)}] \in G^{n*}_{m+2} \leqslant G^{n*}_m.
\end{aligned}$$

□

Lemma 2.2. *a) Let H and K be two arbitrary groups, then for any positive integer m and n,*

$$H^{n*}_m \times K^{n*}_m \subseteq (H \times K)^{n*}_m.$$

b) If H and K be finite groups such that $(|H|, |K|) = 1$, Then for any positive integer m and n,

$$H^{n*}_m \times K^{n*}_m = (H \times K)^{n*}_m.$$

Proof. a) Because $IA(H \times K) = IA(H) \times IA(K)$ and

$$H^{**} \times K^{**} = H' \times K' = (H \times K)' = (H \times K)^{**},$$

by induction on m, it is easy to check that

$$([h,\alpha_1^n,\dots,\alpha_m^n],[k,\beta_1^n,\dots,\beta_m^n]) = [(h,k),\alpha_1^n\times\beta_1^n,\dots,\alpha_m^n\times\beta_m^n],$$

for $\alpha_i \in IA(H)$, $\beta_i \in IA(K)$, $h \in H$ and $k \in K$.

b) It is sufficient to prove that

$$(H\times K)_m^{n*} \subseteq H_m^{n*}\times K_m^{n*}.$$

It is easy to check that $\sigma|_H \in IA(H)$ and $\sigma|_K \in IA(K)$ for all $\sigma \in IA(H\times K)$. Now, by induction on m and n, we have

$$[(h,k),\sigma_1^n,\dots,\sigma_m^n] = ([h,\sigma_1^n|_H,\dots,\sigma_m^n|_H],[k,\sigma_1^n|_K,\dots,\sigma_m^n|_K]),$$

for all $h \in H$, $k \in K$ and $\sigma_1,\dots,\sigma_m \in IA(H\times K)$. This implies the result. □

Lemma 2.3. *If H is a characteristic subgroup of index two of a given group G, then G_m^{n*} is contained in H, for every positive integer m and n.*

Proof. It follows from lemma 2.4 [6]. □

3 n-IA-nilpotent groups

In introduction, we introduced autonilpotent and G^{**}-autonilpotent groups. In this section, we define n-IA-nilpotent groups and study properties of them.

Definition 3.1. A group G is said to be n-IA-nilpotent group of class at most n if the series (2) ends in the identity subgroup after a finite number of steps.

Remark 3.2. The autonilpotent groups are n-IA-nilpotent, but the converse is not true in general. For example, all of abelian groups are n-IA-nilpotent, but

$$G = \bigoplus_{i=1}^{l}\mathbb{Z}_{2^{m_i}},\qquad l>1,\ \ m_1=m_2\geq\dots\geq m_l$$

is not autonilpotent, because $K_n(G)=G$.

Proposition 3.3. *If H or K is not n-IA-nilpotent group, then $H\times K$ is not n-IA-nilpotent.*

Proof. It is clear by lemma 2.2. □

Corollary 3.4. *If H_1, H_2, ..., H_l are n-IA-nilpotent groups with coprime orders, then $H_1\times H_2\times\dots\times H_l$ is also n-IA-nilpotent.*

Proof. The result is follow by induction on l. □

Theorem 3.5. *For a characteristic subgroup H of a given group G, if H and G/H are n-IA-nilpotent, then G is also n-IA-nilpotent.*

Proof. Suppose that there exist positive integers i and j such that

$$H_i^{n*} = \langle 1 \rangle, \qquad \left(\frac{G}{H}\right)_j^{n*} = \langle 1 \rangle.$$

It is clear that $G^{n*}H/H = (G/H)^{n*}$ and by induction on j one gets

$$\frac{G_j^{n*}H}{H} \subseteq \left(\frac{G}{H}\right)_j^{n*} = 1_{\frac{G}{H}}$$

and hence $G_j^{n*} \subseteq H$. Let $[g, \alpha^n]$ be an arbitrary generator of $G_{j+1}^{n*} = [G_j^{n*}, IA(G)]$. It is easy to see that $[g, \alpha^n|_H] \in [H, IA(H)]$, thus $G_{j+1}^{n*} \subseteq H^{n*}$. By induction on i, we have $G_{i+j}^{n*} \subseteq H_i^{n*} = \langle 1 \rangle$. Therefore, G is a n-IA-nilpotent group of class at most i+j. □

Theorem 3.6. *Let H be a proper characteristic subgroup of a given group G with G/H is n-IA-nilpotent of class c. If $H \cap G_c^{n*} = \langle 1 \rangle$, then G is n-IA-nilpotent.*

Proof. Similar to the argument in the proof of theorem 3.5, it is easy to see that $G_c^{n*}H/H \subseteq (G/H)_c^{n*}$. As $H \cap G_c^{n*} = \langle 1 \rangle$, we have

$$\frac{G_c^{n*}}{H \cap G_c^{n*}} \cong \frac{G_c^{n*}H}{H} \subseteq \left(\frac{G}{H}\right)_c^{n*} = 1_{\frac{G}{H}}.$$

Thus $G_c^{n*} = \langle 1 \rangle$ which gives the n-IA-nilpotency of G. □

Theorem 3.7. *For a characteristic subgroup H of a n-IA-nilpotent group G of class c, if $G = HG^{n*}$, then G=H.*

Proof. By hypothesis

$$G^{n*} = [G, IA(G)] = [HG^{n*}, IA(G)].$$

We prove that

$$[HG^{n*}, IA(G)] \leqslant [H, IA(G)]^{G^{n*}}[G^{n*}, IA(G)].$$

Let $h \in H$, $g \in G^{n*}$ and $\alpha \in IA(G)$, then $[hg, \alpha^n] \in [HG^{n*}, IA(G)]$ and

$$\begin{aligned}[hg, \alpha^n] &= (hg)^{-1}\alpha^n(hg) \\ &= g^{-1}h^{-1}\alpha^n(h)\alpha^n(g) \\ &= g^{-1}h^{-1}\alpha^n(h)gg^{-1}\alpha^n(g) \\ &= [h, \alpha^n]^g[g, \alpha^n] \in [H, IA(G)]^{G^{n*}}[G^{n*}, IA(G)].\end{aligned}$$

Because H is a characteristic subgroup, we have

$$\begin{aligned}G^{n*} =&[HG^{n*}, IA(G)] \\ \leqslant&[H, IA(G)]^{G^{n*}}[G^{n*}, IA(G)] \\ \leqslant&HG_2^{n*}.\end{aligned}$$

Thus

$$G = HG^{n*} \leqslant HG_2^{n*}.$$

So, $G = HG_2^{n*}$. Now, by induction we have $G = HG_c^{n*}$, and G=H since G is a n-IA-nilpotent group of class c. □

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