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Italian domination number upon vertex and edge removal

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Abstract

For a graph $G = (V, E)$, *italian domination function* $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either v is adjacent to a vertex assigned 2 under f, or v is adjacent to at least two vertices assigned 1 under f. The *weight* of an italian domination function is the sum of its function values over all vertices. The *italian domination number* $\gamma_I(G)$ equals the minimum weight of an italian dominating function on G. In this paper, we consider the effects of vertex and edge removal on the Italian domination number of a graph. In addition, we characterize the family of cartesian product of some graphs in terms of Italian domination number.

Keywords: edge removal, italian domination number, italian dominating function, graph, vertex removal.

AMS Mathematical Subject Classification [2010]: 05C69

1 Introduction

Throughout this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V, E). For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. Denote by P_n and C_n the path and cycle on n vertices, respectively. The corona of a graph H, denoted $cor(H)$ or $H \circ K_1$ in the literature, is the graph obtained from H by adding a pendant edge to each vertex of H.

Definition 1.1. A set $S \subseteq V$ in a graph G is a *dominating set* if every vertex of G is either in S or adjacent to a vertex of S. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G.

For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [8], [9].

Definition 1.2. A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G.

[1]speaker

Roman domination was introduced by Cockayne et al. in [6] and was inspired by the work of ReVelle and Rosing [12] and Stewart [13]. It is worth mentioning that since 2004, more than hundred papers have been published on this topic, where several new variations were introduced: weak Roman domination [10], maximal Roman domination [2], mixed Roman domination [1], double Roman domination [3], independent Roman domination [7] and also Roman $\{2\}$-domination introduced by [4].

Definition 1.3. For a graph $G = (V, E)$, *italian domination function* (IDF) $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either v is adjacent to a vertex assigned 2 under f, or v is adjacent to least two vertices assigned 1 under f. The *weight* of an italian domination function is the sum of its function values over all vertices. The *italian domination number* $\gamma_I(G)$ equals the minimum weight of a italian dominating function on G.

The concept of italian domination in graphs was introduced by [4] where it was called Roman $\{2\}$-domination. For a graph G, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let $V_i = \{v \in V | f(v) = i\}$ for $i \in \{0, 1, 2\}$. In the whole paper, the function f can be represented by $f = (V_0, V_1, V_2)$.

For many graph parameters, criticality is a fundamental question. The concept of criticality with respect to various operations on graphs has been studied for several domination parameters. Much has been written about graphs where a parameter increases or decreases whenever an edge or vertex is removed or added. This concept has been considered for several domination parameters such as domination number, total domination number, global domination number, secure domination number and Roman domination number, by several authors. This concept is now well studied in domination theory. For references on the criticality concept on various domination parameters see, for example [11]. In this paper we consider this concept for italian domination number.

Definition 1.4. We call a graph G:

- italian domination vertex-critical, or just γ_I-vertex critical, if $\gamma_I(G - v) < \gamma_I(G)$ for each vertex $v \in V(G)$,
- italian domination edge super critical, or just γ_I-edge super critical, if $\gamma_I(G - e) > \gamma_I(G)$ for each edge $e \in E(G)$.

In this paper, we consider the effects of vertex and edge removal on the Italian domination number of a graph. In addition, we characterize the family of cartesian product of some graphs in terms of Italian domination number.

2 Vertex and edge removal

In this section we study vertex removal in italian domination number in graphs. We start with the following lemma.

Lemma 2.1. For any vertex v in a graph G, $\gamma_I(G) \le \gamma_I(G - v) + 1$.

Proof. Suppose that $v \in V(G)$ and f' is a $\gamma_I(G - v)$-function. Then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v) = 1$ and $g(u) = f'(u)$ for $u \in V(G) - \{v\}$, is an IDF of G, and so the result holds. □

Example 2.2. The graph $G = cor(C_4)$ demonstrates that the bound in Lemma 2.1 are sharp.

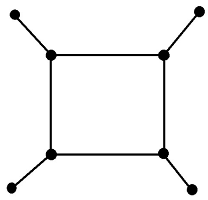

Figure 1: The corona of a graph C_4,

By Lemma 2.1, we have the following Corollary.

Corollary 2.3. For any vertex v in a γ_I-vertex critical graph G, $\gamma_I(G-v) = \gamma_I(G) - 1$.

Proposition 2.4. Let G be a graph of order $n \geq 2$, and let $f = (V_0, V_1, V_2)$ be a γ_I- function. If $v \in V_0$, then $\gamma_I(G-v) = \gamma_I(G)$.

Proof. If we define $g = (V_0 - \{v\}, V_1, V_2)$, then g is an IDF on $G - v$, and thus $\gamma_I(G-v) \leq \omega(g) = |V_1| + 2|V_2| = \gamma_I(G)$. □

Theorem 2.5. A connected unicyclic graph G is γ_I-vertex critical if and only if $G = cor(C_n)$ with $n \equiv 1 \pmod 3$.

Theorem 2.6. For any edge e in a graph G, $\gamma_I(G) \leq \gamma_I(G-e) \leq \gamma_I(G) + 1$.

Proof. Suppose that $e \in E(G)$. Since any γ_I-function for $G - e$ is also an IDF for G, the lower bound is obvious. Let $e = ab$ and f be a $\gamma_I(G)$-function. If $f(a) = f(b) = 0$, or $f(a) > 0$ and $f(b) > 0$, then f is an IDF for $G - e$, and so $\gamma_I(G-e) \leq \gamma_I(G)$. Assume, without loss of generality, that $f(a) = 0$ and $f(b) > 0$. Then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(a) = 1$ and $g(x) = f(x)$ for $x \in V(G) - \{a\}$ is an IDF for $G - e$ and so the upper bound follows. □

Proposition 2.7. *[4]* For the classes of paths P_n and cycles C_n,

$$\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil, \quad \text{and} \quad \gamma_I(C_n) = \lceil \frac{n}{2} \rceil.$$

The following proposition provides examples of γ_I-edge super critical graphs.

Proposition 2.8. (i) The path P_n is γ_I-edge super critical if and only $n \equiv 1 \pmod 2$.

(ii) The cycle C_n is γ_I-edge super critical if and only if $n \equiv 0 \pmod 2$.

3 Italian domination Number of cartesian product in graphs

In this section, we characterize the family of cartesian product of some graphs in terms of Italian domination number.

Definition 3.1. The Cartesian product of graphs G and H is the graph $G\square H$ with vertex set $G \times H$ and $(x_1, x_2)(y_1, y_2) \in E(G\square H)$ whenever $x_1y_1 \in E(G)$ and $x_2 = y_2$, or $x_2y_2 \in E(H)$ and $x_1 = y_1$.

The Cartesian product is commutative and associative, having the trivial graph as a unit (cf. [13]).

Proposition 3.2. *If G is a connected graph of order n and maximum degree $\Delta(G) = \Delta$, then*

$$\gamma_I(G) \geq \frac{2n}{\Delta + 1}.$$

Theorem 3.3. *For $m, n \geq 1$, we have $\gamma_I(C_{5m} \Box C_{5n}) = 10mn$.*

Proof. The lower bound follows from Proposition 3.2. Let $V(C_{5m} \Box C_{5n}) = \{v_{ij} : 0 \leq i \leq 5m-1, 0 \leq j \leq 5n-1\}$. Define $f : V(G) \to \{0, 1, 2\}$ by

$$f(v) = \begin{cases} 2 & \text{if } v \in \{v_{(5i)(5j+2)}, v_{(5i+1)(5j)}, v_{(5i+2)(5j+3)}, v_{(5i+3)(5j+1)}, v_{(5i+4)(5j+4)}, 0 \leq i \leq m-1, 0 \leq j \leq n-1\} \\ 0 & \text{Otherwise.} \end{cases}$$

We can see that $\gamma_I(G) \leq \omega(f) = 10mm$. This complete the proof. □

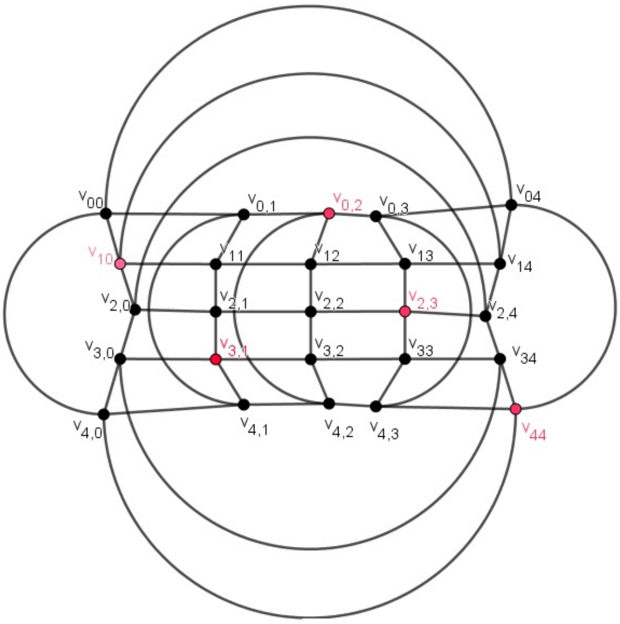

Figure 2: The cartesian product of C_5 and C_5

References

[1] H. Abdollahzadeh Ahangar, T.W. Haynes and J.C. Valenzuela-Tripodoro, *Mixed Roman domination in graphs*, Bull. Malays. Math. Sci. Soc. **93**(7) (2015) 1093–1102.

[2] H. Abdollahzadeh Ahangar, A. Bahremandpour, S.M. Sheikholeslami, N.D. Soner, Z. Tahmasbzadehbaee and L. Volkmann, *Maximal Roman domination numbers in graphs*, Util. Math. **103** (2017) 245–258.

[3] R.A. Beeler, T.W. Haynes and S.T. Hedetniemi, *Double Roman domination*, Discrete Appl. Math. **211** (2016) 23–29.

[4] M. Chellali, T. Haynes, S.T. Hedetniemi, A. McRae, *Roman* $\{2\}$*-domination*, Discrete Appl. Math. **204** (2016) 22–28.

[5] E. W. Chambers, B. Kinnersley, N. Prince, and D. B. West *Extremal Problems for Roman Domination*, **23** (2009) 1575–1586.

[6] E.J. Cockayne, P.A. Dreyer Jr., S.M. Hedetniemic and S.T. Hedetniemic, *Roman domination in graphs*, Discrete Math. **278** (2004) 11-22.

[7] M. Adabi, E. Ebrahimi Targhi, N. Jafari Rad, M. Saied Moradi, *Properties of independent Roman domination in graphs*, Australas. J. Combin. **52** (2012) 11–18.

[8] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

[9] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York (1998).

[10] M.A. Henning and S.T. Hedetniemi, *Defending the roman empire–a new strategy*, Discrete Math. **266** (2003) 239–251.

[11] N. Jafari Rad and L. Volkmann, *Changing and unchanging the Roman domination number of a graph*, Util. Math., **89**, (2012), 79–95.

[12] C.S. Revelle and K.E. Rosing, *Defendens imperium romanum: a classical problem in military strategy*, Amer. Math. Monthly **107** (7) (2000) 585–594.

[13] I. Stewart, *Defend the Roman Empire*, Sci. Amer. **281** (6) (1999) 136-139.