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Palindromes of k-bonacci words on infinite alphabet

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Abstract

The Fibonacci word W on an infinite alphabet was introduced in [Zhang et al., Electronic J. Combinatorics 2017 24(2), 2-52] as a fixed point of the morphism $2i \to (2i)(2i+1)$, $(2i+1) \to (2i+2)$, $i \geq 0$. Here we consider the finite and infinite k-bonacci words on infinite alphabet, these words are denoted by $W_n^{(k)}$ and $W^{(k)}$, respectively. We first consider some properties of these words. Then we count all palindrome factors $W_n^{(k)}$, when n is smaller than k.

Keywords: factor of word, k-bonacci words, palindrome factors.

1 Introduction

Finite and infinite Fibonacci words are among the most studied ones in combinatorics of words and have important roles in computer science, based on their optimal properties and various applications, see for example [5]. The sequence of finite Fibonacci words $(F_n)_{n\geq -1}$ is given by $F_{-1} = 1, F_0 = 0$ and the recurrence relation $F_n = F_{n-1}F_{n-2}$ which holds for $n \geq 1$. An equivalent way to give these words for $n \geq 0$, is using $F_n = \psi^n(0)$, where ψ is the binary morphism $0 \to 01$, $1 \to 0$. The infinite Fibonacci word is then given by $F_\infty = \lim_{n\to\infty} F_n$ or equivalently by $F_\infty = \psi^\omega(0)$.

A natural extension of finite Fibonacci words to k-letter alphabet, $k > 2$, is defining finite k-bonacci words $(F_n^{(k)})_{n\geq 0}$ by

$$F_n^{(k)} = \begin{cases} 0 & \text{if } \ n = 0, \\ F_{n-1}^{(k)} \dots F_0^{(k)} n & \text{if } \ 1 \leq n < k, \\ F_{n-1}^{(k)} \dots F_{n-k}^{(k)} & \text{if } \ n \geq k. \end{cases}$$

Alternatively, these words may be given by $F_n^{(k)} = \psi_k^n(0)$, for $n \geq 0$, where $\psi_k : \{0, \dots, k-1\}^* \to \{0, \dots, k-1\}^*$ is the morphism

$$\psi_k(i) = \begin{cases} 0(i+1) & \text{if } \ i = 0, \cdots, k-2, \\ 0 & \text{if } \ i = k-1. \end{cases} \tag{1}$$

The infinite k-bonacci word is then given by $F_\infty^{(k)} = \lim_{n\to\infty} F_n^{(k)}$ or equivalently by $F_\infty^{(k)} = \psi_k^\omega(0)$.

The infinite Fibonacci word over the infinite alphabet of nonnegative integers, $\mathbb{N}$, denoted here as $W^{(2)}$, is presented in [6] as the fixed point of the morphism φ_2 starting with 0, where φ_2 is given by $\varphi_2(2i) = (2i)(2i+1)$ and $\varphi_2(2i+1) = 2i+2$ for $i \geq 0$. More precisely, we have $W^{(2)} = \varphi_2^\omega(0)$. The authors of [6] have

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also studied the finite Fibonacci words over $\mathbb{N}$, namely the words $W_n^{(2)} = \varphi_2^n(0)$. It is obvious that when the digits of $W_n^{(2)}$ and $W^{(2)}$ are calculated mod 2, these words are reduced to F_n and F_∞, respectively. Among several properties of words $W_n^{(2)}$ and $W^{(2)}$ studied in [6], the authors characterized palindromic factors of $W_n^{(2)}$ and $W^{(2)}$. Particularly, the authors showed that in contrast to the ordinary infinite Fibonacci word which contains palindromic factors of arbitrary length, the word $W^{(2)}$ has no palindrome of length greater than 3. Some more properties of these words were consequently studied by Glen et al. in [4]. Among other results, they computed the number of palindromes in $W_n^{(2)}$.

In this paper, first we study some properties of the k-bonacci words on infinite alphabet. Then, we compute the number maximal palindromic factors of finite k-bonacci words of small order with respect to k.

2 Definitions and notation

In this paper, the alphabet, which can be a finite or a countable infinite set, is denoted as $\mathcal{A}$. When the alphabet is infinite, we simply take $\mathcal{A} = \mathbb{N}$. Each element of the alphabet $\mathcal{A}$ is called a *letter*. When $\mathcal{A} = \mathbb{N}$, we equivalently use the term *digit* instead of letter. We denote by $\mathcal{A}^*$ the set of finite words over $\mathcal{A}$ and we let $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\epsilon\}$, where ϵ is the empty word. We denote by $\mathcal{A}^\omega$ the set of all infinite words over $\mathcal{A}$ and we let $\mathcal{A}^\infty = \mathcal{A}^* \cup \mathcal{A}^\omega$. If $a \in \mathcal{A}$ and $W \in \mathcal{A}^\infty$, then the symbols $|W|$ and $|W|_a$ denote respectively the length of W, and the number of occurrences of letter a in W (It is obvious that when $W \in \mathcal{A}^\omega$, $|W| = \infty$). For any word $W \in \mathcal{A}^\infty$, $\mathcal{A}lph(W)$ is defined to be the set of letters which have at least one occurrence in W, that is $\mathcal{A}lph(W) = \{a \in \mathcal{A} : |W|_a > 0\}$.

A word $V \in \mathcal{A}^*$ is a factor of a word $W \in \mathcal{A}^\infty$, denoted as $V \prec W$, if there exist $U \in \mathcal{A}^*$ and $U' \in \mathcal{A}^\infty$, such that $W = UVU'$. A word $V \in \mathcal{A}^*$ (resp. $V \in \mathcal{A}^\infty$) is said to be a *prefix* (resp. *suffix*) of a word $W \in \mathcal{A}^\infty$, denoted as $V \lhd W$ (resp. $V \rhd W$), if there exists $U \in \mathcal{A}^\infty$ (resp. $U \in \mathcal{A}^*$) such that $W = VU$ (resp. $W = UV$). We denote the prefix (resp. suffix) V of length j of $W \in A^+$ by $\mathrm{Pref}_j(W)$ (resp. $\mathrm{Suff}_j(W)$). If $W \in \mathcal{A}^*$ and $W = VU$ (resp. $W = UV$,) we merely write $V = WU^{-1}$ (resp. $V = U^{-1}W$). For a finite word $W = w_1w_2 \dots w_n$, with $w_i \in \mathcal{A}$ and for $1 \le j \le j' \le n$, we denote $W[j, j'] = w_j \dots w_{j'}$, and for simplicity we denote $W[j, j]$ by $W[j]$. The *reversal* of a finite word $W = w_1w_2 \dots w_n$, with $w_i \in \mathcal{A}$, is $W^R = w_nw_{n-1} \dots w_1$. A word $W \in \mathcal{A}^*$ is called *palindrome* if $W = W^R$. The set of all palindromic factors of the word $W \in \mathcal{A}^\infty$ is denoted by $\mathcal{P}al(W)$. When the alphabet is finite, for any word $U \in \mathcal{A}^\infty$, the number of palindromic factors of length n of U, called the *palindrome complexity* of U, is denoted by $\mathrm{pal}_U(n)$ (for more information about the palindrome complexity see [1, 2] and the references therein). When the alphabet is infinite (i.e. $\mathcal{A} = \mathbb{N}$), the definition of palindrome complexity can naturally be extended to all words $U \in \mathcal{A}^\infty$ with the same notation.

For $1 \le i \le n$, let $U_i \in \mathcal{A}^*$; then $\prod_{i=n}^{1} U_i$ is defined to be $U_nU_{n-1} \dots U_1$. For a finite word W and an integer n, $n \oplus W$ denotes the word obtained by adding n to each letter of W. For example, let $W = 01023$ and $n = 5$, then $n \oplus W = 56578$. For a finite set $S = \{S_1, \dots, S_m\} \subset \mathcal{A}^+$, we define $n \oplus S$ to be the set $\{n \oplus S_1, \dots, n \oplus S_m\}$.

For any integer $k \ge 2$ the sequence of k-bonacci numbers, denoted by $\{f_n^{(k)}\}_{n \ge 0}$, is given as

$$f_n^{(k)} = \begin{cases} 0 & \text{if } n = 0, \dots, k-2, \\ 1 & \text{if } n = k-1, \\ \sum_{i=n-k}^{n-1} f_i^{(k)} & \text{if } n \ge k. \end{cases} \quad (2)$$

The last recurrence relation which holds eventually, states that the n-th term of the sequence is the summation of the k previous ones. This reminds the Fibonacci and Tribonacci recurrence relations in the special cases $k = 2$ and $k = 3$. In fact, $f_n^{(2)}$ and $f_n^{(3)}$ are essentially the well-known Finonacci and Tribonacci numbers.

We define the finite (resp. infinite) k-bonacci words $W_n^{(k)}$ (resp. $W^{(k)}$) on the infinite alphabet $\mathbb{N}$, using the morphism φ_k given below for integers $i \geq 0$ and $0 \leq j < k$,

$$\varphi_k(ki+j) = \begin{cases} (ki)(ki+j+1) & \text{if } j = 0, \ldots, k-2 \\ (ki+j+1) & \text{if } j = k-1. \end{cases}$$

More precisely, $W_n^{(k)} = \varphi_k^n(0)$ and $W^{(k)} = \varphi_k^\omega(0)$ (Note that $W_0^{(k)} = F_0^{(k)} = 0$). Consequently $F_n^{(k)} = W_n^{(k)} \mod k$, that is for a fixed value of k, the k-bonacci words over the infinite alphabet are reduced to k-bonacci words over a finite alphabet when the digits are calculated $\mod k$. It can be shown that for $n \geq 0$,

$$|F_n^{(k)}| = |W_n^{(k)}| = f_{n+k}^{(k)}. \tag{3}$$

3 Some properties of $W_n^{(k)}$

In this section, we give some recursive identities which state the word $W_n^{(k)}$ as the concatenation of previous words of the same type. These identities will help us to discover the structure of palindromes in k-bonacci words in the future sections. First we present a simple lemma stating a property of the morphism φ_k which can be easily deducted from the definition. THe proof of these lemmas can be found in [3].

Lemma 3.1. *For any integer $i \geq 0$, $\varphi_k(k+i) = k \oplus \varphi_k(i)$.*

Lemma 3.2. *For $1 \leq n \leq k-1$,*

$$W_n^{(k)} = \prod_{i=n-1}^{0} W_i^{(k)}\ n. \tag{4}$$

Lemma 3.3. *For $1 \leq n \leq k-1$,*

$$W_n^{(k)} = W_{n-1}^{(k)} W_{n-1}^{(k)} (n-1)^{-1} n.$$

In the next lemma we give a recursive formula for $W_n^{(k)}$ when $n \geq k$.

Lemma 3.4. *For $n \geq k$,*

$$W_n^{(k)} = \prod_{i=n-1}^{n-k+1} W_i^{(k)}\ (k \oplus W_{n-k}^{(k)}). \tag{5}$$

Corollary 3.5. *For integers $0 \leq i \leq n$, $W_i^{(k)}$ is a prefix of $W_n^{(k)}$.*

According to the properties of $W_n^{(k)}$, which was proved in [3] all factors of $W_n^{(k)}$ are divided into three classes of factors, called included factors, bordering factors and straddling factors. The definition of these factors are give below:

Definition 3.6. Let n be a positive integer. Considering factorizations of $W_n^{(k)}$ given in the Equations (4) and (5), we divide the set of factors of $W_n^{(k)}$ into three following types:
Included factors. The digit n or the factors of $W_n^{(k)}$ which are included in any of the words $W_{n-1}^{(k)}$, $W_{n-2}^{(k)}$, …, $W_{(n-k+1)_*}^{(k)}$ or in $(k \oplus W_{n-k}^{(k)})$, if $n \geq k$;

Bordering factors. Factors F which are of the form $F = X_jY_j$ for some $(n-k+1)_*+1 \leq j \leq n-1$, where $X_j \neq \epsilon$ is a suffix of $W_j^{(k)}$ and $Y_j \neq \epsilon$ is a prefix of $\prod_{i=j-1}^{(n-k+1)_*} W_i^{(k)}$. We call any such factor, a *bordering factor of type* j of $W_n^{(k)}$;

Straddling factors. Factors F which are of the form $F = AB$, where $A \neq \epsilon$ is a suffix of $\prod_{i=n-1}^{(n-k+1)_*} W_i^{(k)}$ and $B = n$ if $n \leq k-1$, and B is a prefix of $k \oplus W_{n-k}^{(k)}$ if $n \geq k$; these are called (A,B)*-straddling factors* (or straddling factors for short, if there is no danger of confusion) of $W_n^{(k)}$.

Definition 3.7. Let n be a positive integer. Considering Definition 3.6, we call a palindromic factor P of $W_n^{(k)}$ an *included* (resp. a *bordering*, a *straddling*) *palindrome* if P is an included (resp. a bordering, a straddling) factor of $W_n^{(k)}$.

4 The number of palindromes in $W_n^{(k)}$

In this section, we are going to count the total number of palindromes in $W_n^{(k)}$, when n is smaller than $k-1$. Let $P^{(k)}(n)$ denote the total number of palindromes in $W_n^{(k)}$ occurring in different positions and $B^{(k)}(n,j)$ and $S^{(k)}(n)$ denote the number of bordering palindromes of type j and straddling palindromes of $W_n^{(k)}$, respectively. Then by Definition 3.7, the following recurrence relation holds

$$P^{(k)}(n) = \Sigma_{i=n-4}^{n-1} P^{(k)}(i) + \Sigma_{i=n-2}^{n-1} B^{(k)}(n,i) + S^{(k)}(n). \tag{6}$$

4.1 Palindromes in $W_n^{(k)}$ when $n \leq k-1$

Lemma 4.1. *For* $1 \leq n \leq k-1$, $W_n^{(k)}\, n^{-1}$ *is a palindromic word.*

Proof. We prove this by induction on n. Since for every $k > 2$ we have $W_1^{(k)} = 01$, the first step of the induction is true. Suppose $n = j < k-1$, the word $W_j^{(k)}\, j^{-1}$ is a palindrome. Now using Lemma 3.3, we have

$$\begin{aligned} W_{j+1}^{(k)}(j+1)^{-1} &= W_j^{(k)} W_j^{(k)} j^{-1} \\ &= W_j^{(k)} j^{-1} j W_j^{(k)} j^{-1}, \end{aligned}$$

which is a palindromic word by induction hypothesis. □

Lemma 4.2. *For every* $1 \leq n \leq k-1$, $P^{(k)}(n) = 2P^{(k)}(n-1) + 2^{n-1} - 1$ *and* $P^{(k)}(0) = 1$.

Proof. Since the digit n just occurs in the last position of $W_n^{(k)}$, every palindromic factor of $W_n^{(k)}$ either equals to n or is a palindromic factor of $W_n^{(k)}\, n^{-1}$. By Lemma 3.3, for every $n \leq k-1$, we have

$$W_n^{(k)} n^{-1} = W_{n-1}^{(k)} W_{n-1}^{(k)} (n-1)^{-1} = W_{n-1}^{(k)} (n-1)^{-1} (n-1) W_{n-1}^{(k)} (n-1)^{-1}. \tag{7}$$

From Equation (7), we conclude that $n-1$ occurs once in $W_n^{(k)}\, n^{-1}$. Using Lemma 4.1 and again using Equation (7), we find that a factor P of $W_n^{(k)}\, n^{-1}$ is a palindromic word if and only if it is either a palindromic factor of $W_{n-1}^{(k)}(n-1)^{-1}$ or $P = a(n-1)a$, where a is a prefix of $W_{n-1}^{(k)}$. Therefore, $P^{(k)}(n) = 2(P^{(k)}(n-1) - 1) + |W_{n-1}^{(k)}| + 1$. Hence, we provide $P^{(k)}(n) = 2P^{(k)}(n-1) + 2^{n-1} - 1$. □

Theorem 4.3. *For every* $0 \leq n \leq k-1$, $P^{(k)}(n) = n2^{n-1} + 1$.

Proof. The proof is easy by induction on n. □

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