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Some family of graphs satisfying an extremal identity

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Abstract

Let $G = (V, E)$ be a simple and undirected graph with vertex set V and the edge set E. A subset D of V is called a $[1, k]$-set if every vertex $v \in V \setminus D$ satisfies $1 \leq |N(v) \cap D| \leq k$, where $N(v)$ is the set of neighbors of v in G. In other words every vertex of G has at least one neighbor and at most k neighbors in D, which says that D is a domination set with an additional constrain. The smallest number of a $[1, k]$-set in G is denoted by $\gamma_{[1,k]}(G)$. By the definition it is easy to see that the inequality $\gamma(G) \leq \gamma_{[1,k]}(G)$ always hold (where $\gamma(G)$ stands for the domination number of G). It is proved that deciding whether the equality $\gamma(G) = \gamma_{[1,k]}(G)$ hold, is an NP-hard problem. So, it is worthwhile to construct family of graphs which satisfy the mentioned identity. In this paper we consider some family of graphs which satisfy this identity.

Keywords: Domination number, $[1, k]$-domination number, Interval graphs

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1 Introduction

Let $G = (V, E)$ be a graph. A *dominating set* for G is a subset S of vertices such that each vertex not in S is adjacent to at least one vertex in S. The size of the smallest dominating set is denoted by $\gamma(G)$, and called the *dominating number* of G. The books [5] and [6] survey results on the subject of dominating sets. A generalization of this notion is the notion of a $[j, k]$-*dominating set* first defined by Chellali et al. in [3]: A $[j, k]$-dominating set is a subset S of vertices such that each vertex not in S is adjacent to at least j and at most k vertices in S. The size of the smallest $[j, k]$-dominating set is denoted by $\gamma_{[j,k]}(G)$.

In Question 8 of [3], the authors ask to characterize graphs for which $\gamma(G) = \gamma_{[1,2]}(G)$. In this paper, we consider the more general question of finding when $\gamma(G) = \gamma_{[1,k]}(G)$, i.e., when there exists an optimal (smallest) dominating set such that each vertex not in the dominating set is adjacent to at most k vertices in the dominating set.

In [4], the author showed that for each k, it is computationally hard to test whether $\gamma(G) = \gamma_{[1,k]}(G)$, for a given a graph G. Specifically, they showed that checking whether $\gamma(G) = \gamma_{[1,k]}(G)$ is NP-hard, and hence not possible to do in polynomial time unless $P = NP$. This shows that an efficient characterization of graphs such that $\gamma(G) = \gamma_{[1,k]}(G)$ is probably impossible. Hence, it is worthwhile to sole this problem for special family of graphs. Here, we consider the problem whether $\gamma(G) = \gamma_{[1,k]}(G)$, Next, we show that for

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several classes of graphs, we have $\gamma(G) = \gamma_{[1,k]}(G)$. A set A of vertices of G is an *asteroidal set* if for each vertex $a \in A$, when we remove a and its neighbors from the graph G, the set $A \setminus \{a\}$ is contained in one connected component of the remaining graph. The *asteroidal number* of G, denoted $AT(G)$ is the maximum cardinality of an asteroidal set of G.

2 Definitions and Preliminaries

Definition 2.1. For a vertex v in a graph G, we define the open neighbourhood $N_G(v)$ to be the set of neighbours v, and the closed neighbourhood $N_G[v]$ to be $N_G(v) \cup \{v\}$.

Definition 2.2. A vertex v belonging to a set S is said to have a *private neighbour* if there exists a vertex $u \in N_G(v) \setminus S$ such that u is not adjacent to any other member of S.

Definition 2.3. [2]: A *permutation graph* is built as follows: let L_t and L_b be two parallel lines on the plane (we imagine L_t on the top and L_b on the bottom). Let $x_1, x_2, \cdots, x_n$ be n distinct points on L_t and $y_1, y_2, \cdots, y_n$ respectively be n distinct points on L_b, noting that these points do not necessarily appear on the two lines from left to right in this order. A permutation graph then has the n line segments $[x_i, y_i]$ of the plane as vertices, and two vertices (i.e. segments) are adjacent if the corresponding segments intersect. In other words, given a permutation σ on n elements, if we consider $x_1, \dots x_n = [1, n]$ and $y_1, \dots, y_n = \sigma(1), \dots \sigma(n)$. A permutation graph is just the intersection graph of the lines $[i, \sigma(i)]$. A *bipartite permutation graph* is a permutation graph which is also bipartite. Given a vertex $v = [x_i, y_i]$ we may write $t(v) = x_i$ and $b(v) = y_i$.

Our terminology for permutation graphs comes from [2], where in the literature the term ”permutation graph” is used in other senses as well, e.g. a Cayley graph on the set of all permutations or a generalized prism.

3 Main results

In this section we find the minimum integer k for which $\gamma_{[1,k]}(G) = \gamma(G)$, when G belongs to the family of integral graphs or permutation graphs. We start with the following lemma,which is useful in some proofs regarding the existence of a private neighbour for each element of a minimum dominating set.

Lemma 3.1. *[1] Let G be a graph with no isolated vertices. Then G contains an optimal dominating set S such that each vertex in S has a private neighbour.*

3.1 Permutation graphs

In [4], the author proved that $\gamma_{[1,3]} = \gamma$ for permutation graphs. To prove this, they first proved that in fact any minimum dominating set of a permutation graph is also a $[1, 4]$-dominating set. Then they showed that at least one such dominating set is also a $[1, 3]$-dominating set. Here we show that any bipartite permutation graph satisfy the stronger identity $\gamma_{[1,2]} = \gamma$.

Proposition 3.2. *For every bipartite permutation graph G, $\gamma(G) = \gamma_{[1,2]}(G)$.*

Proof. Let D be an optimal dominating set for which every vertex in it has a private neighbour in $V(G) \setminus D$ (by Lemma 3.1, there exists such an optimal dominating set). We prove that D is a $\gamma_{[1,2]}$-set. Recall that vertices of G are line segments $[x_i, y_i](1 \leq i \leq |V(G)|)$, where $x_i \in L_t$ and $y_i \in L_b$. Assume by contradiction that D is not a $\gamma_{[1,2]}$-set. This means that there is a vertex $v = [x, y] \in V(G) \setminus D$ and three vertices $v_1 = [x_1, y_1], v_2 = [x_2, y_2], v_3 = [x_3, y_3]$ of D, where $x_1 < x_2 < x_3$, such that $v_i \in N(v), 1 \leq i \leq 3$. Since G is bipartite non of the vertices v_1, v_2, v_3 are adjacent. Therefore, from the inequality $x_1 < x_2 < x_3$ we have $y_1 < y_2 < y_3$. Since v is adjacent to all the vertices v_1, v_2, v_3, without loss of generality, we may assume that $y_3 < y$ and $x < x_1$.

Now, every private neighbour v_4 of v_2 should be adjacent to v, so v_4, v_2 and v make a triangle, which is a contradiction by the fact that G is bipartite. Hence, v_2 could not have any private neighbour, which is a contradiction. Hence D is a $[1, 2]$-set, as desired.

□

3.2 Interval graphs

Definition 3.3. An *interval graph* is a graph where each vertex is an (open) interval of a given line, and two vertices are adjacent if they intersect. If furthermore all intervals are of unit length then the graph is called a *unit interval graph.*

In this section we want to prove that for every interval graph G, $\gamma_{[1,3]}(G) = \gamma(G)$. Moreover we give an example of an interval graph G for which the stronger equality $\gamma_{[1,2]}(G) = \gamma(G)$ does not hold. In addition we prove that for every unit interval graph G we have $\gamma_{[1,2]}(G) = \gamma(G)$.

Theorem 3.4. *Let G be an interval graph, then $\gamma(G) = \gamma_{[1,3]}(G)$. Moreover there is an interval graph H for which $\gamma(H) < \gamma_{[1,2]}(H)$.*

Proof. We prove that for every interval graph G, every optimal dominating set is a $\gamma_{[1,3]}$-set. For the contrary suppose that D is an optimal dominating set and there exist a vertex $v \in V(G) \setminus D$ and vertices s_1, s_2, s_3, s_4 in D, which all are adjacent to v, and are ordered increasingly according to their left endpoints. For $1 \leq i \leq 4$, let a_i (resp. b_i) be the left (resp. right) endpoint of s_i. Let a (resp. b) be the left (resp. right) endpoint of v. Since a_1, a_2, a_3, a_4 are ordered increasingly and s_1, s_2, s_3, s_4 is contained in an optimal dominating set we conclude that b_1, b_2, b_3, b_4 are also ordered increasingly (because otherwise one interval for example $s_i \in D$ will included in another interval for example $s_j \in D$ $(j \neq i)$ which means that $N_G[s_i] \subseteq N_G[s_j]$. Hence, $D \setminus \{s_i\}$ is also a dominating set, which is a contradiction). Since v adjacent to all s_1, s_2, s_3, s_4, we have $a < b_1$ and $a_4 < b$. By these facts, it is easy to see that

$$N_G[\{s_1, s_2, s_3, s_4\}] \subseteq N_G[\{s_1, v, s_4\}].$$

This shows that $(D \cup \{v\}) \setminus \{s_2, s_3\}$ is also a domination set, whose cardinality is less than $|D|$, which is an contradiction. So, for every interval graph G, every optimal dominating set D is also a $\gamma_{[1,3]}$-set.

For the second part of the theorem consider the following example. Let H be an interval graph with vertex set

$$V(H) = \{(1, 7), (2, 3), (3, 4), (4, 5), (5, 13), (6, 14), (8, 9), (9, 10), (10, 11), (12, 19), (15, 16), (16, 17), (17, 18)\}$$

Then it is easy to see that H has two optimal dominating set D_1 and D_2, where $D_1 = \{(1, 7), (5, 13), (12, 19)\}$ and $D_2 = \{(1, 7), (6, 14), (12, 19)\}$. None of them is a $[1, 2]$-set, since the vertex $(5, 13)$ has three neighbours in D_2 and the vertex $(6, 14)$ has three neighbours in D_1. Therefore $\gamma(H) < \gamma_{[1,2]}(H)$. □

Theorem 3.5. *For every unit interval graph G, $\gamma(G) = \gamma_{[1,2]}(G)$.*

Proof. Let D be a optimal dominating set. We prove that D is $\gamma_{[1,2]}$-set. For the contrary suppose that there exist s_1, s_2, s_3 in D and $v \in V(G) \setminus D$ such that v adjacent to all s_1, s_2, s_3. Let $\{a_1, a_2, a_3\}$ ($\{b_1, b_2, b_3\}$) be the set of left endpoints (right endpoints) of s_1, s_2, s_3 and suppose that $\{a_1, a_2, a_3\}$ are ordered increasingly, therefore $\{b_1, b_2, b_3\}$ are also ordered increasingly. If a and b denote the left and right endpoints of v, respectively, then $a < b_1$ and $a_3 < b$. Therefore, $|a_3 - b_1| < b - a = 1$. Since all intervals have unit length, we conclude that $N_G[s_2] \subseteq N_G(s_1) \cup N_G(s_3)$, which is a contradiction with the assumption that D be a optimal dominating set. □

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