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A special case of partition problem

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ABSTRACT

Assume that A is a set with m elements.

$$\mathrm{A} = \{1,2, \dots, \mathrm{m}\}$$

Also, assume that consider that $\mathrm{A_t}$ be the set of all partitions of A whose number of members in each section is less than or equal to t.
In this case, we intend to calculate the number of members of this set. Also in the special case when $\mathrm{t} = 2$, we show that

$$\mathrm{cardinal}(\mathrm{A_2}) = \mathrm{cardinal}(\mathrm{B_2}) + (m-1)\mathrm{cardinal}(\mathrm{C_2})$$

When:

$$\mathrm{B} = \{1,2, \dots, \mathrm{m}-1\}$$
$$\mathrm{C} = \{1,2, \dots, \mathrm{m}-2\}$$

Also,

$$|\mathrm{A_2}| = \sum_{k=o}^{[\frac{m}{2}]} \frac{\mathrm{m}!}{k!\,(m-2k)!\,2^k}$$

KEYWORDS: Combinatorial method, Permutation, Partition, Factorial.

1 INTRODUCTION

A partition is a way of writing an integer as a sum of positive integers where the order of the addends is not significant, possibly subject to one or more additional constraints. By convention, partitions are normally written from largest to smallest addends. In mathematics, a partition of a set is a grouping of its elements into non-empty subsets, in such a way that every element is included in exactly one subset. Every equivalence relation on a set defines a partition of this set, and every partition defines an equivalence relation. A set equipped with an equivalence relation or a partition is sometimes called a setoid, typically in type theory and proof theory.

Based on the equivalence between geometric lattices and matroids, this lattice of partitions of a finite set corresponds to a matroid in which the base set of the matroid consists of the atoms of the lattice, namely, the partitions with singleton sets and one two-element set. These atomic partitions correspond one-for-one with the edges of a complete graph. The matroid closure of a set of atomic partitions is the finest common coarsening of them all; in graph-theoretic terms, it is the partition of the vertices of the complete graph into the connected components of the subgraph formed by the given set of edges. In this way, the lattice of partitions corresponds to the lattice of flats of the graphic matroid of the complete graph.

Partitions can be graphically visualized with Young diagrams or Ferrers diagrams. They occur in a number of branches of mathematics and physics, including the study of symmetric polynomials and of the symmetric group and in group representation theory in general. There are two common diagrammatic methods to represent partitions: as Ferrers diagrams, named after Norman Macleod Ferrers, and as Young diagrams, named after the British mathematician Alfred Young. Both have several possible conventions; here, we use English notation, with diagrams aligned in the upper-left corner.The asymptotic growth rate for $p(n)$ is given by:

$$\log\big(p(n)\big) \sim C\sqrt{n} \ \ as\ n \to \infty$$

Where,

$$C = \pi\sqrt{\frac{2}{3}}.$$

The more precise asymptotic formula (see [1]):

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) as\ n \to \infty,$$

was first obtained by G. H. Hardy and Ramanujan in 1918 and independently by J. V. Uspensky in 1920. A complete asymptotic expansion was given in 1937 by Hans Rademacher. If A is a set of natural numbers, we let $p_A(n)$ denote the number of partitions of n into elements of A. If A possesses positive natural density α then

$$\log p_A(n) \sim C\sqrt{\alpha n} \quad \text{as} \ \ n \to \infty.$$

and conversely if this asymptotic property holds for $p_A(n)$ then A has natural density α. This result was stated, with a sketch of proof, by Erdős in 1942. (See [2] and [3])

In this context, the previous work we did in this field can be expressed as the following theorem.

Theorem A. Assume that A is a set with M elements. Also, consider that we intend to divide this set into p partitions. Some subsets may even be empty. Therefore, the value of mathematical expectation of members of the largest subset is equal to:

$$(\sum_{i=1}^{M} \sum_{j=1}^{p} (-1)^{j+1} \binom{p}{j} \frac{\binom{M+p-1-ij}{p-1}}{\binom{M+p-1}{p-1}}) - 1.$$

2 MAIN RESULTS

Main Theorem. Assume that A is a set with m elements.

$$A = \{1,2, \dots, m\}$$

Also, assume that consider that A_t be the set of all partitions of A whose number of members in each section is less than or equal to t.

In this case, we intend to calculate the number of members of this set. Also in the special case when $t = 2$, we show that

$$\text{cardinal}(A_2) = \text{cardinal}(B_2) + (m-1)\text{cardinal}(C_2)$$

When:

$$B = \{1,2,\dots,m-1\}$$
$$C = \{1,2,\dots,m-2\}$$

Also,

$$|A_2| = \sum_{k=o}^{[\frac{m}{2}]} \frac{m!}{k!\,(m-2k)!\,2^k}$$

Proof. First assume that

$$t = 2, m = 3.$$

Therefore,

$$A_2 = \{(1,1,1),(1,2)\}.$$

Thus, we have

$$|A_2| = \binom{3}{0} + \binom{3}{0}\binom{3}{1} = 1 + 3 = 4$$

In another special case assume that:

$$t = 2, m = 6.$$
$$A_2 = \{(1,1,1,1,1,1),(1,1,1,1,2),(1,1,2,2),(2,2,2)\}.$$

Thus, we have

$$|A_2| = \binom{6}{0} + \binom{6}{0}\binom{6}{2} + \binom{6}{0}\binom{6}{2}\binom{4}{2} + 1*5*3*1 = 1 + 15 + 45 + 15 = 76$$

With this preliminary discussion, we have:

$$\text{cardinal}(A_2) = \text{cardinal}(B_2) + (m-1)\text{cardinal}(C_2)$$

When:

$$B = \{1,2,\dots,m-1\}$$
$$C = \{1,2,\dots,m-2\}$$

In fact, we do two ways to prove the above. First, we separate a member. Let's assume that this member is going to be placed alone in a single member subset. Now we look at the problem with a different perspective. We separate a member again and assume that this member is going to be placed in a 2-member set as a subset.

Of course, this selected member should be a permanent member. For example, the number should be one. In fact, once we assume that the number is placed alone in a subset, and once again we assume that the number one is placed in a two-member subset.

Actually, with this discussion, we reach a restored relationship, which was one of our desired goals. We can have another attitude to solve this problem. In fact, it is sufficient to count the number of two-membered subsets of a set, on the condition that some members of the set are in single-member subsets.

Suppose we have a set of m members and we want to count the number of subsets of which k are two-membered and m-2k are single-membered subsets. The number of such subsets is equal to:

$$\frac{\mathrm{m}!}{k!\,(m-2k)!\,2^{k}}$$

In fact, the idea of calculating the above formula is that we can choose the members in pairs. Now, considering the permutation within each category, we will have two pairs and k permutation:

$$\frac{1}{2^{k}}\times\frac{1}{k!}\times\frac{\mathrm{m}!}{(m-2k)!}=\frac{\mathrm{m}!}{k!\,(m-2k)!\,2^{k}}$$

Therefore,

$$|\mathrm{A}_2\ |=\sum_{k=o}^{[\frac{m}{2}]}\frac{\mathrm{m}!}{k!\,(m-2k)!\,2^{k}}$$

But when the number of members of the subsets is more than two, the problem becomes a little more complicated and getting an explicit formula or even a return relationship for this problem does not seem simple.

According to the reliable references that exist about generating functions, it can be shown that:

$$|\mathrm{A}_{\mathrm{t}}\ |=e^{(x+\frac{x^{2}}{2}+\cdots+\frac{x^{t}}{t})}$$

Readers can refer to the references according to the complexity of proving this matter.

□

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