Bounds on resolvent Estrada index of graphs

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Abstract

The resolvent Estrada index of a graph G is defined as $EE_r(G) = \sum_{i=1}^{n}(1 - \frac{\lambda_i}{n-1})^{-1}$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the eigenvalues of the adjacency matrix of G. In this work, we establish some new bounds for this invariant of graphs.

Keywords: Resolvent Estrada index, adjacency matrix, spectral moment, eigenvalues.

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1 Introduction

Let G be a finite simple graph with vertex set $V(G) = \{v_1, \cdots, v_n\}$. The adjacency matrix $\mathbf{A} = \mathbf{A}(G) = [a_{ij}]$ of G is a matrix whose (i, j)-th entry is equal to 1 if vertices i and j are adjacent, and 0 otherwise. The set of all eigenvalues of $\mathbf{A}(G)$ are denoted by $Spec(G) = \{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. The resolvent Estrada index of G was put forward by Estrada and Higham [6] as $EE_r(G) = \sum_{i=1}^{n}(1 - \frac{\lambda_i}{n-1})^{-1}$. As well known, all eigenvalues of G are not more than $n-1$ and the largest eigenvalue λ_1 of G is equal to $n-1$ if and only if G is complete graph K_n. Thus, for all non-complete graphs of order n, $|\frac{\lambda_i}{n-1}| < 1$. Consequently, for all non-complete graphs, due to the Taylor series, we have:

$$EE_r(G) = \sum_{k \geq 0} \frac{M_k(G)}{(n-1)^k}, \tag{1}$$

where $M_k(G)$ is the $k-$th spectral moment of G, i.e. $M_k(G) = \sum_{i=1}^{n} \lambda_i^k$. It is well known that $M_0(G) = n$, $M_1(G) = 0$, $M_2(G) = 2m$, $M_3(G) = 6\,t$ and M_k is equal to the number of closed walks of length k in G [4, 5].

In [1], Chen and Qian proved that if G is a non-complete graph and $e \in E(G)$, then $EE_r(G) > EE_r(G - e)$. As an immediate consequence, the graph $K_n - e$ and the empty graph $\overline{K_n}$ have maximal and minimal resolvent Estrada index, respectively. The authors of this paper also determined the first

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thirteen trees with the greatest resolvent Estrada index, and characterize the multipartite graphs having the maximal resolvent Estrada index. The extremal trees with respect to EE_r-values are investigated in [1, 7] and several bounds for this quantity in terms of the number of vertices and edges are presented in [3, 8]. On the other hand, Gutman et al. [8] found trees, unicyclic, bicyclic, and tricyclic graphs with minimum and maximum values of EE_r.
In this work, we pursue the development of the work done on this issue. More precisely, we give several bounds for this quantity, and specially for the cases of trees and bipartite graphs.

2 Main results

For a simple graph G of order n and size m, the first and second Zagreb indices are defined as $Zg_1(G) = \sum_{v\in V(G)} d^2(v)$ and $Zg_2(G) = \sum_{e=uv\in E(G)} d(u)d(v)$ respectively, where $d(u)$ denotes the degree of vertex u [9]. In this section, some upper bounds for resolvent Estrada index of graphs are presented. For the sake of completeness we mention here a result of Chen and Qian [2] which is crucial for the main result of this section.

Lemma 2.1. *Let $M_k(G)$ be the k-th spectral moment of G with degree sequence $(d_1, d_2, \ldots, d_n)$ and maximum degree Δ. Then*

1. $M_k(G) \leq n\,\Delta^{k-1}$, *for each* $k \geq 2$.
2. $M_k(G) \leq 2\,m\,\Delta^{k-2}$, *for each* $k \geq 2$.
3. $M_k(G) \leq Zg_1(G)\,\Delta^{k-3}$, *for each* $k \geq 3$.
4. $M_k(G) \leq 2\,Zg_2(G)\,\Delta^{k-4}$, *for each* $k \geq 4$.
5. $M_k(G) \leq \sum_{i=1}^{n} d_i^{k-1}$, *for each* $k \geq 2$.

Each of the equalities holds in (1)-(5) if and only if k is even and each component of G is the complete bipartite graph $K_{\Delta,\Delta}$.

Lemma 2.2. *Let G be a graph with with degree sequence $(d_1, d_2, \ldots, d_n)$ and $\Delta < n-1$. Then*

1. $EE_r(G) < n\left(1 + \frac{\Delta}{(n-1)(n-1-\Delta)}\right)$.
2. $EE_r(G) < n + \frac{2m}{(n-1)(n-1-\Delta)}$.
3. $EE_r(G) < n + \frac{2\,m}{(n-1)^2} + \frac{Zg_1(G)}{(n-1)^2(n-1-\Delta)}$.
4. $EE_r(G) < n + \frac{2\,m}{(n-1)^2} + \frac{6\,t}{(n-1)^3} + \frac{2\,Zg_2(G)}{(n-1)^3(n-1-\Delta)}$.
5. $EE_r(G) < n + \frac{1}{n-1}\sum_{i=1}^{n} \frac{d_i}{n-1-d_i} = n - \frac{n}{n-1} + \sum_{i=1}^{n} \frac{1}{n-1-d_i}$.

Since the star graph S_n is the unique $n-$vertex bipartite graph with $\Delta = n-1$, all other bipartite graphs are having $\Delta < n-1$. In the following result, some upper bounds for EE_r of such graphs are presented.

Lemma 2.3. *Let G be a bipartite graph such that $\Delta < n-1$. Then,*

1. $EE_r(G) \leq n\left(1 + \frac{\Delta}{(n-1)^2-\Delta^2}\right)$,

2. $EE_r(G) \leq n + \frac{2m}{(n-1)^2-\Delta^2}$,

3. $EE_r(G) \leq n + \frac{2m}{(n-1)^2} + \frac{\Delta Zg_1(G)}{(n-1)^4-\Delta^2(n-1)^2}$,

4. $EE_r(G) \leq n + \frac{2m}{(n-1)^2} + \frac{2Zg_2(G)}{(n-1)^4-\Delta^2(n-1)^2}$,

5. $EE_r(G) \leq n + \sum_{i=1}^{n} \frac{d_i}{(n-1)^2-d_i^2}$.

In each part, the equality is satisfied if and only if all components of G are the complete bipartite graph $K_{\Delta,\Delta}$.

If G is connected then in each part of Lemma 2.3, equality is satisfied if and only if n is even and $G \cong K_{\frac{n}{2},\frac{n}{2}}$. On the other hand, all inequalities of Lemma 2.3 are strict, when G is acyclic. In [2], it is shown that if G is a tree (or a forest) with $\Delta \geq 2$, then for any $k \geq 3$,

$$M_k(G) < n \left(\sqrt{4(\Delta-1)}\right)^{k-1}. \tag{2}$$

So, (2) holds for $k = 1, 2$.

Lemma 2.4. *Let G be a tree (or a forest) with* $\Delta \geq 2$. *Then,*

$$EE_r(G) < n \left(1 + \frac{n\sqrt{4(\Delta-1)}}{(n-1)^2 - 4(\Delta-1)}\right).$$

Up to now, many lower and upper bounds for the largest and least eigenvalues of graphs were given. In [10, 12], bounds on the spectral radius $\lambda_1(G)$ in terms of n and m of a connected graph G are investigated as:

$$\frac{2m}{n} \leq \sqrt{\frac{Zg_1(G)}{n}} \leq \lambda_1(G) \leq min\{\Delta, \sqrt{n-1}\} \tag{3}$$

Also, for the least eigenvalue $\lambda_n(G)$ are the following [11]:

$$-\frac{n}{2} \leq \lambda_n(G) < -\frac{1+\sqrt{1+\frac{4(n-3)}{n-1}}}{2} \tag{4}$$

Cvetković in [4], demonstrated that the empty graph $\overline{K_n}$ is the only graph with exactly one eigenvalue, and also proved that graph G has spectrum $\{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}\}$, $\lambda_1 > \lambda_2$, if and only if G is the direct sum of m_1 complete graphs of order $\lambda_1 + 1$. Hence, It is straightforward to see that a connected graph with exactly two distinct eigenvalues is complete graph K_n. In the following next three Lemmas, we obtain several upper bounds for resolvent Estrada index of graphs by applying the above relations and the inequalities stated in the last section.

Lemma 2.5. *Let G be a non-complete connected graph or empty graph with n vertices, then*

$$EE_r(G) \leq [\frac{n^2}{4}] \frac{(\lambda_1-\lambda_n)^2}{n(n-1-\lambda_1)(n-1-\lambda_n)} + n$$

with equality if and only if $G \cong \overline{K_n}$.

Lemma 2.6. *Let G be a non-complete connected graph or empty graph with n vertices, then*

(1). $EE_r(G) \leq \frac{n(n-1)(n-1-\lambda_1-\lambda_n)}{(n-1-\lambda_1)(n-1-\lambda_n)}$

(2). $EE_r(G) \leq \frac{n(2n-2-\lambda_1-\lambda_n)^2}{4(n-1-\lambda_1)(n-1-\lambda_n)}$

(3). $EE_r(G) \leq \frac{n(n-1)}{n-1-\left(\sqrt{n-1-\lambda_n}-\sqrt{n-1-\lambda_1}\right)^2}$

(4). $EE_r(G) \leq \left(\sqrt{n} + \frac{1}{2\sqrt{n}}.\sqrt{\frac{n-1-\lambda_n}{n-1-\lambda_1}}\left(\sqrt{\frac{n-1-\lambda_n}{n-1-\lambda_1}} - 1\right)^2\right)^2$

(5). $EE_r(G) \leq \frac{n(\lambda_1-\lambda_n)^2}{3(n-1-\lambda_1)(n-1-\lambda_n)} + n$

With equality in each of the above parts, if and only if $G \cong \overline{K_n}$.

Lemma 2.7. *Let* G *be a non-complete connected graph or empty graph with* n *vertices, then*

(1). $EE_r(G) < [\frac{n^2}{4}].\frac{(n+2\gamma)^2}{2n(3n-2)(n-1-\gamma)} + n$

(2). $EE_r(G) < \frac{n(n-1)(3n-2-2\gamma)}{(3n-2)(n-1-\gamma)}$

(3). $EE_r(G) < \frac{n(5n-2\gamma-4)^2}{8(3n-2)(n-1-\gamma)}$

(4). $EE_r(G) < \frac{n(n-1)}{n-1-\left(\sqrt{\frac{3}{2}n-1}-\sqrt{n-1-\gamma}\right)^2}$

(5). $EE_r(G) < \left(\sqrt{n} + \frac{\sqrt{2}}{8\sqrt{n}}.\sqrt{\frac{3n-2}{n-1-\gamma}}\left(\sqrt{\frac{3n-2}{n-1-\gamma}} - \sqrt{2}\right)^2\right)^2$

(6). $EE_r(G) < \frac{n(2\gamma+n)^2}{6(3n-2)(n-1-\gamma)} + n$

Where, $\gamma = min\{\Delta, \sqrt{n-1}\}$

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