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The strong incidence coloring of outerplanar graphs

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Abstract

An incidence in a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident to v. Two incidences (v, e) and (u, f) are adjacent if at least one of the following holds: (i) $v = u$, (ii) $e = f$, or (iii) edge $vu = e$ or f. A strong incidence coloring of a graph G is a mapping from the set of incidences of G to the set of colors $\{1, \ldots, k\}$ such that any two incidences that are adjacent or adjacent to the same incidence receive distinct colors. The minimum number of colors needed for a strong incidence coloring of a graph is called the strong incidence chromatic number. In this paper we prove that strong incidence chromatic number of every outerplanar graph G is at most $4\Delta(G)$ and this bound is tight.

Keywords: Incidence coloring, strong incidence coloring and outerplanar graph

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1 Introduction

All graphs considered in this paper are simple graphs. We denote by $V(G)$ and $E(G)$, the set of vertices and the set of edges of a graph G respectively, by $\Delta(G)$ the maximum degree of G (we will use Δ if no ambiguity). A vertex with degree one and an edge incident to a vertex of degree one is called a leaf and pendant edge respectively.

An incidence of a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident with v. Two incidences (v, e) and (w, f) of G are adjacent whenever (i) $v = w$, or (ii) $e = f$, or (iii) $vw = e$ or f.

For every vertex v in a graph G, we denote by $A^-(v)$ the set of incidences of the form (v, vu) and by $A^+(v)$ the set of incidences of the form (u, uv). Every edge uv of G has two incidences (u, uv) and (v, vu).

A k-incidence coloring of G is a mapping from the set of incidences of G to the set of colors $\{1, \ldots, k\}$ such that every two adjacent incidences receive distinct colors. The smallest k for which G admits a k-incidence coloring is the incidence chromatic number of G, denoted by $\chi_i(G)$. Incidence colorings were first introduced and studied by Brualdi and Quinn Massey [3].

A k-strong edge coloring of G is a mapping from the set of edges of G to the set of colors $\{1, \ldots, k\}$ such that any two edges meeting at a common vertex, or being adjacent to a same edge of G, are assigned

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different colors. The smallest k for which G admits a k-strong edge coloring is the strong chromatic index of G, denoted by $\chi'_s(G)$. The concept of strong edge coloring was introduced by Fouquet and Jolivet [4].

The strong version of incidence coloring is defined in a similar way. A k-strong incidence coloring of a graph G is a mapping from the set of incidences of G to the set of colors $\{1,\dots,k\}$ such that any two incidences that are adjacent or adjacent to the same incidence receive distinct colors. The smallest k for which G admits a k-strong incidence coloring is the strong incidence chromatic number, denoted by $\chi_i^s(G)$. The strong incidence coloring has been introduced by Benmedjdoub and Sopena [1] in 2021. They expressed the following results.

Proposition 1.1. *[1] For every graph G with maximum degree Δ we have*

$$\sigma(G) \leq \chi_i^s(G) \leq 3\Delta^2 - 2\Delta + 1,$$

where $\sigma(G) = \max_{uv\in E(G)}\{2deg_G(v) + deg_G(u) - 1\}$.

Proposition 1.2. *[1] For every graph G, $\chi_i^s(G) \leq 2\chi'_s(G)$.*

In [1] it is proved that for every cycle C_n $(n \geq 3)$, $\chi_i^s(C_n) = \chi(C_{2n}^4)$. So we have

Proposition 1.3. *For every integer $n \geq 3$,*

$$\chi_i^s(C_n) = \begin{cases} 8 & \text{if } n = 4, \\ 5 & \text{if } n \equiv 0 \pmod 5, \\ 7 & \text{if } n = 7, \\ 6 & \text{otherwise.} \end{cases}$$

A block of a graph G is a maximal connected subgraph of G without a cut-vertex. Notice that each block of a connected graph with order at least two is either a maximally 2-connected subgraph or a single edge.

A planar graph is said to be outerplanar if it has a plane embedding such that all vertices lie on the infinite face. A puffer graph is defined in [2] and is a graph obtained by adding some (possibly empty) pendant edges to each vertex of a cycle or adding a common neighbour to two consecutive vertices of the cycle to get an ear of puffer graph (Notice that for an outerplanar graph, at most one such vertex can be added). Two ears are adjacent when they have a common vertex on the cycle. In a drawing of a puffer graph G, we suppose pendant edges lie on the infinite face.

It is easily seen that for a 2-connected outerplanar graph, there is an ear decomposition where each ear contains at least one internal vertex and the endpoints of every ear are adjacent in the preceding graph (if not, the outerplanarity property is affected).

In [5], an upper bound $3\Delta - 3$ was given for the strong edge coloring of general outerplanar graph. So by Proposition 1.2, for any outher planar graph G we have $\chi'_s(G) \leq 6\Delta - 6$. In this work, we improve this upper bound.

2 Main results

First, we obtain a strong incidence colorings of the puffer graphs. Then, we use a decomposition of an outerplanar graph G to the puffer graphs and prove that every outerplanar graph G has a strong incidence coloring with $4\Delta(G)$ colors and this bound is tight.

Lemma 2.1. *For every puffer graph G with maximum degree Δ, we have $\chi_i^s(G) \leq 4\Delta$.*

Now we consider a puffer graph G with maximum degree 3. In this case, we do not have any two adjacent ears. So we have:

Lemma 2.2. *For every puffer graph G with maximum degree 3, $\chi_i^s(G) \leq 11$.*

Now by using Lemma 2.1, we obtain an strong incidence coloring of a 2-connected outerplanar graph G with $4\Delta(G)$ colors.

Theorem 2.3. *Let G be a 2-connected outerplanar graph with maximum degree Δ, we have $\chi_i^s(G) \leq 4\Delta$. Moreover, this upper bound is tight.*

Lemma 2.4. *Let G be a 2-connected outerplanar graph G and G' be a graph obtained by adding some pendant edges to G. Then we have $\chi_i^s(G') \leq 4\Delta(G')$.*

By using Theorem 2.3 and Lemma 2.4 we have:

Theorem 2.5. *For every outerplanar graph G, we have $\chi_i^s(G) \leq 4\Delta$.*

Note that so far we could not find an outerplanar graph except C_4 with maximum degree Δ and strong incidence chromatic number 4Δ, but the graphs $G_i (i \geq 4)$ in the Figure 1, are graphs with maximum degree i and $\chi_i^s(G_i) = 4\Delta(G_i) - 1$. As it can be seen in the Figure 1, the set of incidences $A^-(v_i)$ $(1 \leq i \leq 3)$, $A^+(v_1)$ and (w_1, w_1v_2) are pairwise adjacent or adjacent with the same incidence with the size of $4\Delta(G_i)-1$. So they have to get dictinct colors. Therfore we have $\chi_i^s(G_i) = 4\Delta(G_i) - 1$. For $\Delta = 2, 3$, it is sufficient to

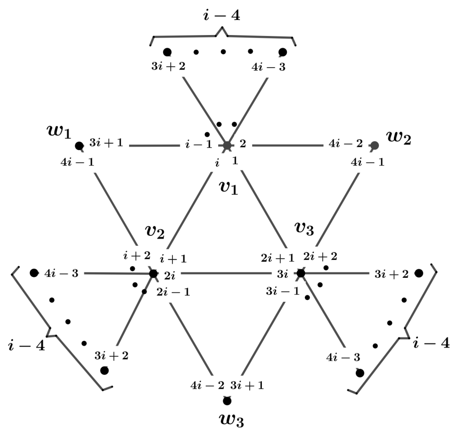

Figure 1: The graph G_i with maximum degree i and $\chi_i^s(G_i) = 4\Delta(G_i) - 1$.

consider graphs G_2 and G_3 in the Figure 2, respectively. We have $\chi_i^s(G_2) = 7, \chi_i^s(G_3) = 11$. Therefore we conjecture the following:

Conjecture 1. For every outerplanar graph G except C_4, we have $\chi_i^s(G) \leq 4\Delta(G) - 1$.

We prove this conjecture for some classes of outerplanar graphs.

Theorem 2.6. *For every outerplanar graph G with maximum degree 3, we have $\chi_i^s(G) \leq 11$. Moreover, this upper bound is tight.*

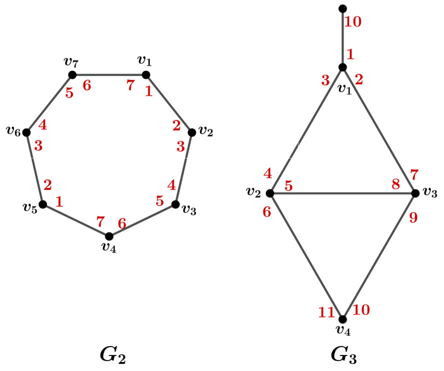

Figure 2: The graphs G_2 and G_3 with $\chi_i^s(G_2) = 7$ and $\chi_i^s(G_3) = 11$.

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