Large -Maximal Small Submodules

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ABSTRACT

In this study, the ideas of Large-maximal small submodules and Large-maximal small Radical of modules are presented. The primary attributes and features of the concept of large-maximal small submodules are provided. Additionally, we discuss the connections between this idea and various submodule kinds with the help of examples and observations that are relevant to our work. Where a proper submodule A of a T-module G is said to be Large-maximal small submodule, if A + B = G where B be a proper submodule of G, then B is Large-Maximal submodule of G. A Large-maximal small Radical of module is sum of all Large-maximal small submodule in G. It will be studied how this concept of radicality relates to other radical conceptions.

KEYWORDS: maximal submodule, Large-maximal submodule, Large-small submodule, Radical of module, Large-small Radical of module.

1 INTRODUCTION

T is a commutative ring with unity in this article. Kasch introduced the concepts of a large (essential) submodule, a small submodule, and a maximal submodule in 1982, respectively. "A proper submodule A of a T-module G is considered, if whenever B is a submodule of G with A < B ≤ G indicates that B = G,[7] "A proper submodule A of an T-module G is called small (A ≪ G), if for any submodule B of G such that A + B = G, indicates that B = G [7], and "A proper submodule A of an T-module G is called Large (essential) submodule in G, (A ≤e G) if for every non zero submodule B of G, then A ∩ B ≠ 0 [7]. The term "uniform module" was first used by Inoue in 1983. He stated that "a T-module G is considered a uniform module if every none zero submodule of G is an essential submodule in G." [12]. The Large-Maximal submodule notion was explained to Amira and Sahira in (2021), " It is referred to as a Large-maximal (L-maximal) submodule of G when it is a proper submodule A of an T-module *G* is said to be Large-maximal (L-maximal) submodule of G if there exists a submodule *B* of *G* such that A < B ≤ G , then *B* is essential submodule of *G* ($B \leq e$ G)[2], who also put forward the idea of the Large-Small Submodule in the same year "A proper submodule A of an T-module G is referred to as Large-small submodule , if A + B = G where B be a submodule of G, then B is essential submodule of G (B ≤e G)[1] The term "closed submodule" was first used by Goodearl in 1976. "A submodule A of G is considered closed in G if it has no appropriate essential extension in G," he wrote [9]. Remember that a T-module G is considered to be multiplication if for each A is a submodule of G, there exists an ideal I of T such that A=IG, Equivalently G is multiplication if for each A≤ G, A = $(A:_T G)G$, where $(A:_T G)=\{t\in T: tG \subseteq A\}$, [13]. If Ann(G) = 0, G is referred to as a faithful module, [10]. The study of various generalizations of small submodules has piqued the interest of numerous scholars; see [5], [6].

As a generalization of Large-maximal submodule and Large-small submodule, the notion of Large-maximal small submodule is introduced in this study.

The various qualities, traits, theorems, and examples of this type of submodule are given in section one. The concept of Large-maximal tiny Radical of G, together with its features and connections to Radical of G, are described in Section two.

2 Large-Maximal Small Submodules

The terms large-maximal small submodules and large-maximal small radical of modules are introduced in this section. By using examples and observations, we highlight key attributes and aspects of these notions.

***Definition* 2.1**

Let G be a T-module and A be a proper submod of G, A is called large-maximal small submodule (in short L-max-s submod) denoted by A $\ll_{L-max}$ G if A+B=G, where B a proper submod of G, then B is L-max submod of G.

An ideal H of T is called large-maximal small ideal (in short L-max-s ideal) of T denoted by H $\ll_{L-max}$ T, if H+K=T for a proper K of T, then K is L-max ideal in T

***Example* 2.2**: Consider Z as Z module, 2Z and 3Z are L-max-s submod since 3Z+2Z=Z where 2Z and 3Z is L-max submod since $2Z < Z \leq_e Z, 3Z < Z \leq_e Z$

***Remarks and Examples* 2.3**

1. Every L-max submod is L-max-s submod, but the converse incorrect in general, for example in Z as Z module 9Z is L-max-s submod since 9Z+8Z=Z and 8Z<2Z$\leq_e$Z that is 8Z is L-max submod, but 9Z isn't L-max since 9Z<3Z$\nleq_e$Z.
2. It's clear that every max. submod is L-max-s submod, but the converse is not true in general, for example: in Z as Z submod 4Z is L-max-s submod since 4Z+5Z=Z, 5Z is L-max submod where 5Z<Z$\leq_e$Z, but 4Z isn't max submod in Z.
3. Each small submod is not L-max-s submod and if a submod is L-max-s submod is not small submod, for example: $(\bar{2}), (\bar{3})$ in Z_6 as Z-module are L-max-s submods since $(\bar{2}) + (\bar{3}) = Z_6$, where $(\bar{2}), (\bar{3})$ are L-max submods, but $(\bar{2}), (\bar{3})$ aren't small submods in Z_6.
4. Sum of L-max submods is L-max-s submod.
5. Let G be uniform module, then every submod of G is L-max-s submod of G, but the converse is not true in general, for example: in Z_6 as Z-module $(\bar{2}), (\bar{3})$ are L-max-s submod, but $(\bar{2}), (\bar{3})$ are not essential in Z_6 , so that Z_6 isn't uniform module.
6. Every L-small submod is L-max-s submod.
   Proof: Let A be a proper submod G and A+K=G, then K$<$ G, since A is L-small submod, hence K $<_e$G $\leq_e$G, so that K is L-max submod by [2], thus A is L-max-s submod.
7. Every simple module is uniform module, then by (5), we get every submod is L-max-s
8. If $\frac{G}{A}$ is simple module, then A is max submod by [8], henece by (2), A is L-max-s submod.
9. Every essential submod of *G* is L-maximal submod of G [2], then by (1), we get it is L-max-s submod of G.
10. If $\frac{G}{A}$ is uniform module, then any submod of *G* is L-max-s.

Proof: Let A be a proper submod of G such that A<B≤G, then $\frac{B}{A} \leq \frac{G}{A}$, but $\frac{G}{A}$ is uniform hence $\frac{B}{A} \leq_e \frac{G}{A}$ and thus B $\leq e$ G, so that A is L-max submod, then by (1), we have A is L-max-s submod of G.

11. Let G be semisimple module, then every submod of G is L-max-s.

12. Let G be semisimple module, then every submod of G is L-max-s. if and only if it is max. submod.

13. Every chained module is uniform module, then every submod is L-max-s submod by (5), where an T-module G is called chained if for submod U, V of G, then either $U \leq V$ or $V \leq U$,[4].

14. Every non zero F-regular module has a max submod [11], and hence it is L-max-s submod by (2), where a T-module G *is* called F-regular if every submod of G is pure [11].

15. A submod of L-max-s submod is not necessary L-max-s submod, for example: in Z_{24} as Z-module, $(\bar{2})$, $(\overline{12})$ are proper submods of Z_{24} , $(\overline{12}) < (\bar{2})$, $(\bar{2}) \ll_{L-max} Z_{24}$, where $(\bar{2})$ is L-max-s submod, since $(\bar{2}) + (\bar{3}) = Z_{24}$ and (3) is L-max submod, since $(\bar{3}) < Z_{24} \leq e$ Z_{24}, but $(\overline{12})$ is small submod in Z_{24}, then by (3), we have $(\overline{12})$ is not L-max-s submod.
16. Let A, B be proper submods such that B < A if B $\ll_{L-max}$ A and A $\ll_{L-max} G$, then B is not necessary L-max-s submod in G, for example: in Z_{12} as Z-module, $(\bar{6}) < (\bar{2})$, $(\bar{6}) \ll_{L-max} (\bar{2})$ since $(\bar{6}) + (\bar{4}) = (\bar{2})$, $(\bar{4})$ is L-max in $(\bar{2})$ and $(\bar{2}) \ll_{L-max} Z_{12}$ since $(\bar{2}) + (\bar{3}) = Z_{12}$ where $(\bar{3})$ is L-max in Z_{12}, but $(\bar{6})$ is small in Z_{12}, then by (3), we have is not L-max-s submod of Z_{12}.

Proposition 2.4

Let A and B be L-max-s submods of a T-module G, then $A + B$ is L-max-s submod of G

proof: Since A and B are L-max-s submods of G, then $A + H = G$ and $B + K = G$ where H, K are L-max submods of G, hence $A + B + (H + K) = G$ where $H + K$ is L-max submod of G by [2], so that $A + B$ is L-max-s submod of G

Proposition 2.5

Let A, B be submods of G and $B \leq A$ if $B \ll_{L-max} G$, then $A \ll_{L-max} G$.

Proof: let K be a proper submod of G such that A + K = G, but $B \leq A$ and $B \ll_{L-max} G$, hence B+K=G, then K is L-max submod, thus $A \ll_{L-max} G$

Corollary 2.6

Let A, B be submods of G. If $A \cap B \ll_{L-max} G$, then $A \ll_{L-max} G$ and $B \ll_{L-max} G$.

proof: Since $A \cap B \leq A \leq G$, $A \cap B \leq B \leq G$ and and $A \cap B \ll_{L-max} G$, then by Proposition (2.5), we have $A \ll_{L-max} G$ and $B \ll_{L-max} G$

Remark 2.7

The converse of *Corollary* (2.6), is not true, for example: in Z_6 as Z-module $(\bar{2}), (\bar{3})$ are L-max-s submod, but $(\bar{2}) \cap (\bar{3}) = (\bar{0})$ where $(\bar{0})$ is small in Z_6, then by Remarks and Examples (2.3) part (3), we have $(\bar{0})$ is not L-max-s submod in Z_6.

Proposition 2.8

Let A, B be submods of G and $B \leq A \leq G$ such that B not small submod in G. If $A \ll_{L-max} G$, then $B \ll_{L-max} G$.

Proof: Since B not small submod in G, so that B + K = G for K is a proper submodule of G, but $B \leq A$, hence A+K=G, then K is L-max submod since $A \ll_{L-max} G$, thus $B \ll_{L-max} G$

corollary 2.9

Let A, B be submods of G and $A \ll_{L-max} G$ and $B \ll_{L-max} G$, such that $A \cap B$ not small submod in G. Then $A \cap B \ll_{L-max} G$

proof: Since $A \cap B \leq A \leq M$ such that $A \cap B$ not small submod in G, and $A \ll_{L-max} M$, then by Proposition (2.8), we have $A \cap B \ll_{L-max} G$

Proposition 2.10

Let A, B be submods of a T-module G such that $B \leq A \leq G$ if $B \ll_{L-max} A$ and $A \leq_e G$, then $B \ll_{L-max} G$.

Proof: Let K be a submod of G such that B + K= G, then $(B + K) \cap A = G \cap A = A$, so by modular law, we get $B + (K \cap A) = A$, but $B \ll_{L-max} A$, hence $K \cap A$ L-max in A, so that $K \cap A <_e A \leq_e G$ then $K \cap A \leq_e M$ by [7], but $K \cap A \leq K \leq G$ and $K \cap A$ is L-max submod in G, then $K \leq_e G$, so that K is L-max submod, thus $B \ll_{L-max} G$.

Proposition 2.11

Let A, B be submods of an T-module G such that $B \leq A \leq G$ and $A \leq^{\oplus} G$. If $B \ll_{L-max} G$, then $B \ll_{L-max} A$.

Proof: Since $A \leq^{\oplus} G$, then exists a submod W of G such that $G = A \oplus W$. To prove $B \ll_{L-max} A$, let B+C=A for a proper submod C of A, so that (B+C) +W=G, hence B+(C+W) =G, but $B \ll_{L-max} G$, then C+W is L-max submod in G, that is $C+W < L \leq_e G$, then $(C+W) \cap A < L \cap A \leq_e G \cap A = A$, hence by modular law, we get $C + (W \cap A) < L \cap A \leq_e A$, then $C < L \cap A \leq_e A$ since $W \cap A = 0$, thus C is L-max submod in A, so that $B \ll_{L-max} A$.

Proposition 2.12

Let $f: G_1 \to G_2$ be an epimorphism where G_1, G_2 be T-modules such that $A \ll_{L-max} G_2$, then f^{-1} (A) $\ll_{L-max} G_1$

Proof: let f^{-1} (A) + B = G_1, so that ff^{-1} (A) + f(B) = $f(G_1)$, then A + f(B) = G_2 since f is onto, but $A \ll_{L-max} G_2$, hence f(B) is L-max submod in G_2, so that f^{-1} (f(B) = B is L-max submod in G_1 by [2], thus $f^{-1}(A) \ll_{L-max} G_1$.

Proposition 2.13

Let A, B be submods of G such that $B \leq A \leq G$. If $\frac{A}{B} \ll_{L-max} \frac{G}{B}$, then $A \ll_{L-max} G$.

Proof: Let A+H=G for H is a proper submod of G, hence $\frac{A+H}{B} = \frac{G}{B}$, so that $\frac{A}{B} + \frac{H}{B} = \frac{G}{B}$, but $\frac{A}{B} \ll_{L-max} \frac{G}{B}$, then $\frac{H}{B}$ is L-max submod in $\frac{G}{B}$. To prove H is L-max submodule in G. Let $H < K \leq G$, so that $\frac{H}{B} < \frac{K}{B} \leq \frac{G}{B}$, but $\frac{H}{B}$ is L-max submod, so that $\frac{K}{B} \leq_e \frac{G}{B}$, then $K \leq_e G$, hence H is L-max submod of G, thus $A \ll_{L-max} G$

Proposition 2.14

Let A, B be submods of an T-module G, such that $B \leq A \leq G$ and B is closed submod in G if $A \ll_{L-max} G$, then $\frac{A}{B} \ll_{L-max} \frac{G}{B}$

Proof: let $\frac{A}{B}+\frac{K}{B}=\frac{G}{B}$ for $\frac{K}{B}$ is a proper submod of $\frac{G}{B}$, then $\frac{A+K}{B}=\frac{G}{B}$, hence A+ K =G, but $A \ll_{L-max} G$, so that K is L-max submod in G. To prove $\frac{K}{B}$ is L-max submod in $\frac{G}{B}$, let $\frac{K}{B}<\frac{L}{B}\leq\frac{G}{B}$ for some $\frac{L}{B}$ submod in $\frac{G}{B}$, then K< L $\leq G$, but K is L-max submodule in G, hence L $\leq_e G$,but B closed submod in G, so that $\frac{L}{B} \leq_e \frac{G}{B}$ by [9], hence $\frac{K}{B}$ is L-max submod in $\frac{G}{B}$, thus $\frac{A}{B} \ll_{L-max} \frac{G}{B}$.

Theorem 2.15

Let A, B and C be submods of a T-module such that $B \leq A \leq C \leq G$ such that A is not small submod in G, and A, B are closed submods in G, then $\frac{C}{B} \ll_{L-max} \frac{G}{B}$ if and only if $\frac{C}{A} \ll_{L-max} \frac{G}{A}$ and $\frac{A}{B} \ll_{L-max} \frac{G}{B}$

Proof: ⟹) suppose that $\frac{C}{B} \ll_{L-max} \frac{G}{B}$. To prove $\frac{C}{A} \ll_{L-max} \frac{G}{A}$, let $\frac{C}{A}+\frac{L}{A}=\frac{G}{A}$ for $\frac{L}{A}$ is a proper submod of $\frac{G}{A}$, hence $\frac{C+L}{A}=\frac{G}{A}$, so that C+L= G, then $\frac{C+L}{B}=\frac{G}{B}$, hence $\frac{C}{B}+\frac{L}{B}=\frac{G}{B}$, but $\frac{C}{B} \ll_{L-max} \frac{G}{B}$, then $\frac{L}{B}$ is L-max submod of $\frac{G}{B}$. Now, to prove L is L-max submod, let $L<K\leq G$, hence for some $K \leq G$, hence $\frac{L}{B}<\frac{K}{B}\leq\frac{G}{B}$, but $\frac{L}{B}$ is L-max submod, so that $\frac{K}{B} \leq_e \frac{G}{B}$, hence $K \leq_e G$, then L is L-max submod of G, so that C $\ll_{L-max} G$, since A is closed in G, then by Proposition (2.14), we have $\frac{C}{A} \ll_{L-max} \frac{G}{A}$. Now, to prove $\frac{A}{B} \ll_{L-max} \frac{G}{B}$, since $\frac{C}{B} \ll_{L-max} \frac{G}{B}$, then by Proposition (2.13), we have $C \ll_{L-max} G$ and by Proposition (2.8), we have $A \ll_{L-max} G$, but B is closed submod in G and by Proposition (2.14) we have $\frac{A}{B} \ll_{L-max} \frac{G}{B}$

⟸) let $\frac{C}{A} \ll_{L-max} \frac{G}{A}$, to prove $\frac{C}{B} \ll_{L-max} \frac{G}{B}$ let $\frac{C}{B}+\frac{L}{B}=\frac{G}{B}$ for $\frac{L}{B}$ is a proper submod of $\frac{G}{B}$, then $\frac{C+L}{B}=\frac{G}{B}$, hence $C+L=G$, so that $\frac{C+L}{A}=\frac{G}{A}$, hence $\frac{C}{A}+\frac{L}{A}=\frac{G}{A}$, but $\frac{C}{A} \ll_{L-max} \frac{G}{A}$, then $\frac{L}{A}$ is L-max submod in $\frac{G}{A}$. To prove L is L-max submod in G, let $L<K\leq G$, then $\frac{L}{A}<\frac{K}{A}\leq\frac{G}{A}$, but $\frac{L}{A}$ is L-max submod , hence $\frac{K}{A} \leq_e \frac{G}{A}$, so that $K \leq_e G$, then L is L-max submod, hence $C \ll_{L-max} G$ and since B is closed submod in G, then by Proposition (2.12), we have $\frac{C}{B} \ll_{L-max} \frac{G}{B}$.

Theorem 2.16

Let G be faithful, finitely generated and multiplication T-module and A a proper submod of G, then the following statmentes are equivalent:

1) A is L-max-s submod of G;

2) $(A:_T G)$ is L-max-s ideal of T.

Proof: **(1) → (2)** let $(A:_T G)+J=T$, for J be a proper ideal of T, since G is multiplication module, $(A:_T G)G+JG=TG$, so that $A+JG=G$, but $A \ll_{L-max} G$ then JG is L-max submod of G. To prove J is L-max ideal of T, let $J<I\leq T$ for some I an ideal of T, then $JG<IG\leq G$, but JG is L-max submod, hence $IG \leq_e G$, so that $I \leq_e T$, hence J is L-max ideal of T, thus $(A:_T G)$is L-max-s ideal of T.

(2) ⟹ (1) Let $A+B=G$ for B is a proper submod of G, since G is multiplication module, hence $A=(A:_T G)G$ and $B=(B:_T G)G$ by [13], then $(A:_T G)G+(B:_T G)G=TG$, since G is faithful, finitely generated and multiplication module, then by [3], we have $(A:_T G)+(B:_T G)=T$, but $(A:_T G)$ is L-max-s ideal of T, thus $(B:_T G)$ is L-max ideal of T, to prove $(B:_T G)G$ is L-max submod of G, suppose that $(B:_T G)G<U\leq G$ for some U is a submod of G, since G is multiplication module, so that $U=JG$ for some J an ideal of T, then $(B:_T G)G<JG<TG$, hence $(B:_T G)<J<T$, but $(B:_T G) \ll_{L-max} T$, then $J \leq_e T$, hence $JG \leq_e G$ by [13], so $U \leq_e G$, hence $(B:_T G)G$ is L-max submod, this is mean B is L-max submod, thus $A \ll_{L-max} G$.

Theorem 2.17

Let a T-module G be a faithful, finitely generated and multiplication and let H be an ideal of T, then $H \ll_{L-max} T$ if and only if $HG \ll_{L-max} G$

Proof: ⟹) let $H \ll_{L-max} T$, to prove $HG \ll_{L-max} G$ suppose that $HG + A = G$ for a proper submod of G, but G is multiplication module, then $A = KG$ for some K an ideal of T, hence $HG + KG = TG$ so that $(H + K)G = TG$, but G is faithful, finitely generated and multiplication module, so that $H + K = T$ by [3] and since $H \ll_{L-max} T$, then K is L-max ideal of T, to prove KG is L-max submod in G, let $KG < JG \leq G$ for some J an ideal of T, then $K < J \leq T$, but K is L-max ideal in T, so that $J \leq_e T$, hence $JG \leq_e G$ by [13], then KG is L-max submod in G, hence $HG \ll_{L-max} G$
⟸) Let $HG \ll_{L-max} G$. To prove $H \ll_{L-max} T$, let $H + K = T$ for K is a proper ideal of T, so that $HG + KG = TG$ since G is multiplication module, that is $HG + KG = G$, but $HG \ll_{L-max} G$, hence KG is L-max submod in G, to prove K is L-max ideal of T, let $K < J \leq T$ for some J an ideal of T, hence $KG < JG \leq G$ since G is multiplication module, but KG is L-max submod in G, then $JG \leq_e G$, hence $J \leq_e T$, so that K is L-max ideal in T, thus $H \ll_{L-max} T$.

3 Large- Maximal Small Radical

The principles of the Large-maximal small Radical of G, together with its attributes and connections to the Radical of G, are discussed in this section.

Definition 3.1

If G is a T-module, the L-max small radical of G is the sum of all L-max-s submodules and is represented by $Rad_{L-max-s}$ (G), defined by: $Rad_{L-max-s}(G) = \sum_{A \ll_{L-max} G} A$.

Remember that " Rad (G) is the sum of all small submodule of G" [7], and " $Rad_{L.s}(G)$ is the sum of all Large-small submod of G where $Rad_{L.s}(G) = \sum\{A \leq G \mid A \ll_{L.s} G\}$ "[1]

Remark and examples 3.2

1) $Rad_{L-max-s}(Z_{12}) = Z_{12}$, but $Rad_{L-s}(Z_{12}) = (\bar{3})$ and Rad $(Z_{12}) = (\bar{6})$
2) $Rad_{L-max-s}(Z_6) = Z_6$, but $Rad_{L-s}(Z_6) = (0)$ and Rad $(Z_6) = (0)$
3) $Rad_{L-max-s}(Z) = Z$, but $Rad_{L-s}(Z) = Z$ and Rad $(Z) = (0)$
4) $Rad(G) \leq Rad_{L-s} \leq Rad_{L-max-s}(G)$
5) If G is semisimple, then $Rad_{L-max-s}(G) = G$

proof: Since G is semisimple, then every proper submod is direct sum, and hence by Remarks and Examples (2.3) part (11), every proper submod is L-max-s submod of G, then $Rad_{L-max-s}G = \sum_{A \ll_{L-max} G} A = G$.

Proposition 3.3

Let G be a T-module and A be a submod of G. If $f: G \to A$ is an epimorphism, then $f^{-1}(Rad_{L-max-s}(A)) \leq Rad_{L-max-s}(G)$

Proof: Suppose $Rad_{L-max-s}(A) = \sum_{B \ll_{L-max} A} B$, but $B \ll_{L-max} A$, hence by Proposition (2.12), we have $f^{-1}(B) \ll_{L-max} G$ such that $f^{-1}(B) \leq G$, hence $f^{-1}(Rad_{L-max-s}(A)) \leq Rad_{L-max-s}(G)$.

Proposition 3.4

Let G be a T-module and A be a submod of G, such that $A \leq_e G$, then $Rad_{L-max-s}(A) \leq Rad_{L-max-s}(G)$

Proof: Let B be a submod of $Rad_{L-max-s}(A)$ such that $B \ll_{L-max} A$, but $A \leq_e G$, then by Proposition (2.10), we have $B \ll_{L-max} G$, so that B is submod of $Rad_{L-max-s}(G)$, thus $Rad_{L-max-s}(A) \leq Rad_{L-max-s}(G)$.

Proposition 3.5

Let A_i, $i \in I$ be submods of a T-module G such that $A_i \ll_{L-max} G, \forall i \in I$. Then $\sum_{i\in I} A_i \ll_{L-max} G$ if and only if $Rad_{L-max-s}(G) \ll_{L-max} G$.

Proof: $\Longrightarrow$) Since $Rad_{L-max-s}$ (G) = sum of all L-max-s submod of G and $A_i \ll_{L-max} G$, $\forall i \in I$, so that $Rad_{L-max-s}$ (G) $=\sum_{i\in I} A_i$. but $\sum_{i\in I} A_i \ll_{L-max} G$, then $Rad_{L-max-s}(G) \ll_{L-max} G$.

$\Longleftarrow$)Suppose that $Rad_{L-max-s}(G) \ll_{L-max} G$. Since $A_i \ll_{L-max} G, \forall i \in I$, hence $\sum_{i\in I} A_i \leq Rad_{L-max-s}(G)$, but $Rad_{L-max-s}(G) \ll_{L-max} G$, then $\sum_{i\in I} A_i \ll_{L-max} G$.

3 CONCLUSION

The research concludes every max.(L-max) and L-s submod is L-max-s submod, and if uniform (semisimple or simple) module, then every submod is L-max-s submod. Also, a submod of L-max-s submod is not necessary L-max-s submod see Remarks and Examples (2.3),

The following is the study's some findings:

1)Let A and B are L-max-s submods of a T-module G, then $A + B$ is L-max-s submod of G

2) Let A, B be submods of G and $B \leq A$ if $B \ll_{L-max} G$, then $A \ll_{L-max} G$.

3) Let A, B be submods of G and $B \leq A \leq G$ such that B not small submod in G. If $A \ll_{L-max} G$, then $B \ll_{L-max} G$.

4) Let A, B be submod of a T-module G such that B ≤ $A \leq G$ if B $\ll_{L-max}$ A and A $\leq_e$G, then B $\ll_{L-max}$G

5) Let A, B be submods of a T-module G such that B ≤ A ≤ G and A$\leq^{\oplus} G$. If B $\ll_{L-max}$G, then B $\ll_{L-max}$A.

6) Let A, B be submods of G such that B ≤ A ≤ G. If $\frac{A}{B} \ll_{L-max} \frac{G}{B}$, then $A \ll_{L-max} G$.

7) Let A, B be submods of a T-module G, such that $B \leq A \leq G$ and B is closed submod in G if A $\ll_{L-max} G$, then $\frac{A}{B} \ll_{L-max} \frac{G}{B}$

8) Let A, B and C be submods of a T-module such that $B \leq A \leq C \leq G$ and A, B are closed submods in G, then $\frac{C}{B} \ll_{L-max} \frac{G}{B}$ if and only if $\frac{C}{A} \ll_{L-max} \frac{G}{A}$.

9) Let G be faithful, finitely generated and multiplication T-module and A a proper submod of G, then the following statmentes are equivalent:

i) A is L-max-s submod of G

ii) $(A:_T G)$ is L-max-s ideal of T

10) Let a T-module G be a faithful, finitely generated and multiplication and let H be an ideal of T, then $H \ll_{L-max} T$ if and only if $HG \ll_{L-max} G$

11) $Rad(G) \leq Rad_{L-s} \leq Rad_{L-max-s}(G)$

12) Let G be a T-module and A be a submod of G. If $f: G \to A$ is an epimorphism, then $f^{-1}(Rad_{L-max-s}(A)) \leq Rad_{L-max-s}(G)$

13) Let G be a T-module and A be a submod of G, such that $A \leq_e G$, then $Rad_{L-max-s}(A) \leq Rad_{L-max-s}(G)$

14) Let A_i, $i \in I$ be submods of a T-module G such that $A_i \ll_{L-max} G, \forall i \in I$. Then $\sum_{i\in I} A_i \ll_{L-max} G$ if and only if $Rad_{L-max-s}(G) \ll_{L-max} G$.

REFERENCES

[1] A. A. Abduljaleel and S. M. Yaseen, "On Large-Small submodule and Large-Hollow module", Journal of Physics: Conference Series, vol.1818 no. 012214, pp.1-7, 2021.

[2] A. A. Abduljaleel and S. M. Yaseen, "Large-Maximal submodules", Journal of Physics: Conference Series, vol. 1963, no. 012011, pp. 1-5, 2021.

[3] A.G. Naoum," Cancellation Modules", Kyungpook Mathematical Journal ,vol.36, no. 1, pp. 97-106, 1996.

[4] B.L. Osofsky, "A Construction of Nonstandared Unieserial Modules over Valuation Domain", Bulletin Amer.Math.Soc. vol. 25, pp. 89-97, 1991

[5] D.X. Zhou and X.R. Zhang, "Small-Essential Submodules and Morita Duality",South east Asian Bull. Math., vol. 35, pp. 1051-1062, 2011.

[6] E. Mustafa and W. Khalid, "on generalization of small submodule", Sci.Int. (Lahore), vol. 30, no. 3, pp. 359- 365, 2018.

[7] F. Kasch, "Modules and Rings", Academic Press, Inc-London, 1982.

[8] F.W. Anderson and K. R. Fuller, " Ring and categories of modules", Springer -Veralg, New York, 1992.

[9] K. R. Goodearl, "Ring Theory, Nonsingular Rings and Modules", Marcel Dekker, New York and Basel, 1976.

[10] R. Wisbauer, "Foundations of Modules and Rings Theory", Gordon and Breach, Philadelphia, 1991.

[11] S. M. Yaseen, "F-Regular Modules", M.Sc Thesis, University of Baghdad, Iraq, 1993.

[12] T. Inoue, "Sum of Hollow Modules", Osaka J.Mtah. vol. 20, pp. 331-336, 1983.

[13] Z. A. Elbast and P.P. Smith, "Multiplication Modules", Communications in Algebra, vol. 16, no. 4, pp. 755-779, 1988.