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Soc–T–ABSO Submodules and Related Concepts

Marwah W. Allami , Wafaa H. Hanoon
Dept of Math./College of Edu
Univ. of Kufa, Najaf, Iraq.
marwahw100@gmail.com
wafaah.hannon@uokufa.edu.iq

ABSTRACT

In this paper, *we present the idea of $Soc\text{–}T\text{–}ABSO$ submodule observations and examine how -$T\text{–}ABSO$ submodule and Socle relate to one another in terms of qualities and characteristics. Submodules of several kinds are now introduced. It is possible to develop new concepts based on $Soc\text{–}T\text{–}ABSO$ submodules using the implications of this work.*

KEYWORDS: $T\text{–}ABSO$ submodules***;*** *$Soc\text{–}T\text{–}ABSO$ submodules; $Soc\text{–}T\text{–}ABSO$ ideal; Socle of module**.**.*

1 INTRODUCTION

In this study, V has the identity and would be a commutative ring, and W is a unitary $V\text{-}module$. The idea of prime submodules of modules was researched by Chin-Pi Lu in 1983 [4]. The idea of prime submodules has numerous generalizations, including Muntaha H.Abdul-Raza introduced $Quasi-prime\ modules\ and\ Quasi-prime\ submodules$ in 1999 and researched [6]. M. Behoodi and H. Koohi developed and researched the idea of weakly prime modules [7]. in 2004. The two ideas of 2-Absorbing and Weakly 2-Absorbing Submodules were generalized in 2011 by Ahmad Yousefian and Fatemeh Soheilnia [3]. Abdulrahman Abdullah researched 2-absorbing modules and a few of their generalizations in 2015 [1]. In 2019, Omar A.Abdulla published the Pseudo Quasi-2-Absorbing submodules and Some Related Concepts, which he described as a generalization of the Pseudo2-Absorbing quasi primary submodules *[8].*

There are two parts to this research. We introduce various concepts in the first section, along with some of their fundamental characteristics. The second section examines the connections, characteristics, and fundamental findings of the $Soc\text{–}T\text{–}ABSO$ submodule.

2 PERLIMINARIES

There are numerous fundamental notions, and this section lists their characteristics.

Definition 2.1*[4]*

A proper submodule A of an $V-module$ W is said to be prime submodule if $ax \in A$ for some $a \in V, x \in W$, implies that $x \in A$ or $a \in (A:_V W)$

Definition 2.2*[5]*

An V-module W is said to be semisimple if W is asummation of simple submodule of W.
Besides, the socle of W is sum of simple submodule of W and denoted by Soc(W) and W is called semisimple if W = Soc(W).

For an arbitrary a submodule A of *an* V- *module* W, Soc(A)= $A \cap Soc(W)$

Definition 2.3*[3]*

A proper submodule A of an V- module W is said to be $T-ABSO$ submodule of W if whenever $a, b \in V, x \in W$ such that $abx \in A$, then either $ax \in A$ or $bx \in A$ or $ab \in (A:_V W)$.

Definition 2.4*[6]*

A proper submodule A of an V-module W is said to be quasi-prime submodule if for some $a, b \in V, x \in W$ such that $abx \in A$, then either $ax \in A$ or $bx \in A$.

Definition 2.5 *[5]*

A submodule A of an V-module W is said to be essential in W brief $A \leq_e W$ if for all non-zero submodule B of W, $A \cap B \neq 0$.

Definition 2.6 *[2]*

A proper ideal I of a ring V is said to be $T-ABSO$ *ideal* if *for some* $a, b, c \in V$ with $abc \in I$, then either $ab \in I$ or $bc \in I$ or $ac \in I$.

Definition 2.7 *[5]*

Let A *a proper submodule of an V- module* W

(a) *A proper submodule B of an V- module W* is said to be an addition complement, shortly adco, of A in W *if and only if*
   (1) $A + B = W$.
   (2) B is minimal in $A + B$.

(b) *A a proper submodule C of an V- module W* is said to be an intersection complement, shortly inco, of A in W *if and only if*
   (1) $A \cap C = 0$.
   (2) C is maximal in $A \cap C$.

Definition 2.8 *[5]*

Let $\Lambda = \{B_i | i \in I\}$ be a set of submodules B_i of W then $\sum_{i \in I} B_i = \langle \bigcup_{i \in I} B_i \rangle$ is said to be *the sum of the submodules* $\{B_i | i \in I\}$.

Proposition 2.9 *[5]*

If $f: W \to W'$, f is monomorphism and $Im(f) \leq_e W'$, then $f(Soc(W)) = Soc(W')$ and $Soc(W) = f^{-1}(Soc(W'))$

Corollary 2.10 *[5]*

Let *C, D are a submodules of an $V -$ module W*. Then, we have $C \oplus D = W$ if and only if D is an inco and adco of C in W.

Lemma (MODULAR LAW) 2.11 *[5]*

Let A, B and C be submodules of an V- module W and B≤ C *then*

$(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$.

Definition 2.12*[5]*

(1) A subset D of a module W is called a generating set of W if and only if $\langle D \rangle = W$

(2) An V-module W *is called finitely generated if and only if there exists a finite generating set*

3 $Soc-T-ABSO$ SUBMODULES AND RELATED CONCEPTS

Definition 3.1

A proper submodule A of an V – module W is called Socle –Two– Absorbing (in short is $Soc\!-T\!-ABSO$) submodule of W if wheneve $a, b \in V, x \in W$ such that $abx \in A$, then either $ax \in A + Soc(W)$ or $bx \in A + Soc(W)$ or $ab \in (A + Soc(W) :_V W)$.

Example 3.2

Consider Z_{24} as Z – module, take $A = <\overline{12}>$ such that $Soc(Z_{24}) = <\overline{4}>$ and $<\overline{12}> + <\overline{4}> = <\overline{4}> = A + Soc(Z_{24})$ and $(<\overline{4}> :_Z Z_{24}) = 4Z$.

Then for all $a, b \in Z$ and $x \in Z_{24}$ at least two of a, b, x are even or one of them is 4 then either $ab \in (A + Soc\ (Z_{24}) :_Z Z_{24})$ or $ax \in A + Soc(Z_{24})$ or $bx \in A + Soc(Z_{24})$ then A is $Soc\!-T\!-ABSO$ submodule

Remarks and Example 3.3

1. *It is evident that every $T - ABSO$ submodule is $Soc\!-T\!-ABSO$submodule, but in general the inverse is not true, for example in Z_{24} as Z – module, take $A = <\overline{12}>$ is $Soc\!-T\!-ABSO$ submodule of Z_{24}, but A is not $T\!-ABSO$submodule of Z_{24} , since $2 \cdot 2 \cdot \overline{3} \in A$,then $2 \cdot \overline{3} \notin A$ and $2 \cdot 2 = 4 \notin (A :_Z Z_{24}) = 12Z$.*

2. *The intersection of each pair of distinct prime submodules of an V – module W is $Soc\!-T\!-ABS$submodule.*

   Proof :*Let A, B be two distinct prime submodule of W . Assume that $a, b \in V$ and $x \in W$ such that $abx \in A \cap B$. Suppose that $ax \notin A \cap B + Soc(W)$ and $bx \notin (A \cap B) + Soc(W)$, then $ax \notin A \cap B$ and $bx \notin A \cap B$. The case $ax \notin A$ and $bx \notin A$ this is contridiction since A is prime submodule, similar if $ax \notin B$ and $bx \notin B$ this is contridiction since $B$is prime.The case $ax \notin A$ and $bx \notin B$ Since $abx \in A \cap B$ then $abx \in A$ & $abx \in B$. Then $b \in (A :_V W)$ and $a \in (B :_V W)$, hence $ab \in (A \cap B :_V W)$ and so $abW \subseteq A \cap B \subseteq A \cap B + Soc(W)$, hence $abW \subseteq A \cap B + Soc(W)$, then $ab \in (A \cap B + Soc(W) :_V W)$.Thus $A \cap B$ is $Soc\!-T\!-ABSO$ submodule of W.*

3. *It is evident that every quasi-prime submodule is $Soc\!-T\!-ABSO$ submodule, but in general the inverse is not true, see example in part(1) is $Soc\!-T\!-ABSO$ submodule of Z_{24}, but A is not quasi-prime submodule of Z_{24} , since $2 \cdot 2 \cdot \overline{3} \in A$,then $2 \cdot \overline{3} \notin A$.*

4. *If A , B be two $Soc\!-T\!-ABSO$ submodules of an V – module W and $A \subset B$ if B is $Soc - T - ABSO$ submodule of W then it is not necessary that A is a $Soc\!-T\!-ABSO$ submodule, for example if $W = Z \oplus Z$ as Z-module, $A = <(4,0)>$ and $B = <(2,0)>$, $Soc(Z \oplus Z) = (0,0), B + Soc(Z \oplus Z) = <(2,0)>$, and $(B + Soc(Z \oplus Z) :_Z Z \oplus Z) = (0,0)$ Then if $a, b \in Z$, $x \in Z \oplus Z$ with $abx \in B$ at least one of it (ax or bx is even) i.e either $ax \in B + Soc(W)$ or $bx \in B + Soc(W)$.Then B is $Soc\!-T\!-ABSO$ submodule of W, but A is not $Soc\!-T\!-ABSO$ submodule of Z – module $Z \oplus Z$ since $2 \cdot 2 \cdot (3,0) \in <(4,0)>$ but $2 \cdot (3,0) \notin <(4,0)> + Soc(Z \oplus Z)$ and $2 \cdot 2 = 4 \notin (A + Soc(Z \oplus Z) :_Z Z \oplus Z) = (0,0)$.*

5. *If A, B be two submodule of an V – module W and $A \subset B$ with B is $T\!-ABSO$ submodule of W, if A is a $Soc\!-T\!-ABSO$ of W then A is a $Soc\!-T\!-ABSO$ submodule of B .*

   Proof: *Let $a, b \in V$, $x \in B$, so $x \in W$ such that $abx \in A$, hence $abx \in B$, but A is $Soc - T - ABSO$ submodule of W and B is $T\!-ABSO$ submoduleof W , then $ax \in B$ or $bx \in B$ or $ab \in (B :_V W)$ and $ax \in A + Soc(W)$ or $bx \in A + Soc(W)$ or $abW \subseteq A + Soc(W)$. Then $ax \in (A + Soc(W)) \cap B$ or $bx \in (A + Soc(W)) \cap B$ or $abW \subseteq \left(A + Soc(W)\right) \cap B$.*

   But $\left(A + Soc(W)\right) \cap B = A + (B \cap Soc(W)) = A + Soc(B)$ by(Lemma 2.11 and Defintion(2.2)[5]) hence $ax \in A + Soc(B)$ or $bx \in A + Soc(B)$ or $abB \subseteq A + Soc(B)$.

   Then either $ax \in A + Soc(B)$ or $bx \in A + Soc(B)$ or $ab \in (A + Soc(B) :_V B)$. Thus A is $Soc\!-T\!-ABSO$ submodule of B.

6. *Let A, B be two submodules of an V – module W and $A \subset B$ with $Soc(W) = Soc(B)$. If A is $Soc\!-T\!-ABSO$ submodule of W, then A is $Soc\!-T\!-ABSO$ submodule of B.*

   Proof: *Let $a, b \in V$, $x \in B$ then $x \in W$ such that $abx \in A$, so that $abx \in B$. But A is*

$Soc\text{–}T\text{–}ABSO$ submodule of W, then $ax \in A + Soc(W)$ or $bx \in A + Soc(W)$ or $abW \subseteq A + Soc(W)$, hence $ax \in A + Soc(B)$ or $bx \in A + Soc(B)$ or $abB \subseteq abW \subseteq A + Soc(B)$ Then either $ax \in A + Soc(B)$ or $bx \in A + Soc(B)$ or $ab \in (A + Soc(B) :_V B)$.Thus A is $Soc\text{–}T\text{–}ABSO$ submodule of B.

7. The sum of two $Soc\text{–}T\text{–}ABSO$ submodules is not necessary $Soc\text{–}T\text{–}ABSO$ submodule for example: Let $A = 2Z, B = 3Z$ each A and B is $Soc\text{–}T\text{–}ABSO$ submodules in Z – module Z, but $A + B = 2Z + 3Z = Z$ is not $Soc\text{–}T\text{–}ABSO$ submodule of Z.
8. Let A and B be two submodules of an V – module W such that $A \simeq B$. If A is $Soc\text{–}T\text{–}ABSO$ submodule of W, it does not have B is $Soc\text{–}T\text{–}ABSO$ submodule of W, for example: $A = 2Z$ is $Soc\text{–}T\text{–}ABSO$ submodule of Z – module Z and $2Z \simeq 30Z$, but $30Z$ is not $Soc\text{–}T\text{–}ABSO$ submodule, since $2 \cdot 3 \cdot 5 = 30 \in 30Z$, but $2 \cdot 5 \notin 30Z + Soc(Z) = 30Z$, $3 \cdot 5 \notin 30Z + Soc(Z) = 30Z$ and $2 \cdot 3 \notin (30Z + Soc(Z) :_Z Z) = 30Z$.
9. The intersection of two $Soc\text{–}T\text{–}ABSO$ submodules, it is not necessary that $Soc\text{–}T\text{–}ABSO$ submodule, for example: 3Z and 4Z are $Soc - T - ABSO$ submodules in Z – module Z, but $3Z \cap 4Z = 12Z$ which is not $Soc - T - ABSO$ submodule, since $2 \cdot 2 \cdot 3 = 12 \in 12Z$, then $2 \cdot 3 \notin 12Z + Soc(Z) = 12Z$ and $2 \cdot 2 \notin (12Z + Soc(Z) :_Z Z) = 12Z$.
10. Every finitely generated an R-module $W \neq 0$ has a $Soc\text{–}T\text{–}ABSO$ submodule.
11. Every submodule of semisimple module is a $Soc\text{–}T\text{–}ABSO$

Now, we give theorems which are characterization $Soc\text{–}T\text{–}ABSO$ submodule as follows:

Theorem 3.4.

Let A be a proper submodule of W then the following statements are equivalent :

1. A is $Soc\text{–}T\text{–}ABSO$ submodule of W
2. If $abH \subseteq A$ for som $a, b \in V$ and a submodule H of W implies either $aH \subseteq A + Soc(W)$ or $bH \subseteq A + Soc(W)$ or $ab \in (A + Soc(W) :_V W)$

Proof : $(1) \to (2)$: let A is an $Soc\text{–}T\text{–}ABSO$ submodule of W and let H be a submodule of W. Such that $abH \subseteq A$ for some $a, b \in V$ assume that $aH \nsubseteq A + Soc(W)$ and $bH \nsubseteq A + Soc(W)$ and $ab \notin (A + Soc(W) :_V W)$ then there exist $h_1, h_2 \in H$ with $ah_1 \notin A + Soc(W)$ and $bh_2 \notin A + Soc(W)$ Since $abh_1 \in A$ and $ab \notin (A + Soc(W) :_V W)$ and $ah_1 \notin A + Soc(W)$, we get $bh_1 \in A + Soc(W)$. Also since $abh_2 \in A$ and $ab \notin (A + Soc(W) :_V W)$ and $bh_2 \notin A + Soc(W)$, we get $ah_2 \in A + Soc(W)$. Now, since $ab(h_1 + h_2) \in A$ and $ab \notin (A + Soc(W) :_V W)$, we have $a(h_1 + h_2) \in A + Soc(W)$ or $b(h_1 + h_2) \in A + Soc(W)$ if $a(h_1 + h_2) \in A + Soc(W)$ then $ah_1 + ah_2 \in A + Soc(W)$ since $ah_2 \in A + soc(W)$ then we get $ah_1 \in A + Soc(W)$ wich is contradiction! if $b(h_1 + h_2) \in A + Soc(W)$ then $bh_1 + bh_2 \in A + Soc(W)$ and since $bh_1 \in A + Soc(W)$, we get $bh_2 \in A + Soc(W)$ which is contradiction! therefore $ab \in (A + Soc(W) :_V W)$ or $aH \subseteq A + Soc(W)$ or $bH \subseteq A + Soc(W)$.
$(2) \to (1)$ it is evident .

Theorem 3.5.

A proper submodule A of an V-module W is $Soc\text{–}T\text{–}ABSO$ submodule of W if and only if $IJH \subseteq A$ for some ideals I, J of V and some submodule H of W then either $IH \subseteq A + Soc(W)$ or *$JH \subseteq A + Soc(W)$* or *$IJ \subseteq (A + Soc(W) :_V W)$.*

Proof: $\Longrightarrow$) suppose that A is $Soc\text{–}T\text{–}ABSO$ submodule of W and let I , J are ideals of V and H is a submodule of W, let $i \in I$ and $j \in J$, $ijH \subseteq A$. But A is $Soc\text{–}T\text{–}ABSO$ submodule, then by Theorem(3.4), we have either $iH \subseteq A + Soc(W)$ or $jH \subseteq A + Soc(W)$ or $ijW \subseteq A + Soc(W)$, so that either $IH \subseteq A + Soc(W)$ or $JH \subseteq A + Soc(W)$ or $IJW \subseteq A + Soc(W)$.
$\Longleftarrow$)Let $ijx \in A$ for some $i, j \in V$, $x \in W$, hence $< i >< j >< x > \subseteq A$, then either $< i >< x > \subseteq A + Soc(W)$ or $< j >< x > \subseteq A + Soc(W)$ or $< i >< j > \subseteq (A + Soc(W) :_V W)$, so that either $ix \in A + Soc(W)$ or $jx \in A + Soc(W)$ or $ij \in (A + Soc(W) :_V W)$. Thus A is $Soc\text{–}T\text{–}ABSO$ submodule of W.

Definition 3.6

A proper ideal I of a ring V is called $Soc\text{-}T\text{-}ABSO$ ideal of V if whenever $abc \in I$, where $a,b,c \in V$ implies that either $ab \in I + Soc(V)$ or $bc \in I + Soc(V)$ or $ac \in I + Soc(V)$, where $Soc(V) =$ sum of all simple ideals of V.

Example 3.7

Consider a ring Z if $I = 2Z$ is $Soc\text{-}T\text{-}ABSO$ ideal of Z.

Proposition 3.8

Let A a proper submodule of an V-module W. If A is $T\text{-}ABSO$ submodule of W, then $(A:_V W)$ is $Soc\text{-}T\text{-}ABSO$ ideal of V.

Proof :Let $abc \in (A:_V W)$ for some $a,b,c \in V$. Then $abcW \subseteq A$, hence $ab(cx) \subseteq A$ for all $x \in W$, but A is $T\text{-}ABSO$ submodule of W. Then either $a(cx) \in A$ or $b(cx) \in A$ or $ab \in (A:_V W)$, hence either $a(cW) \subseteq A$ or $b(cW) \subseteq A$ or $ab \in (A:_V W)$, therefore either $ac \in (A:_V W) \subseteq (A:_V W) + Soc(V)$ or $bc \in (A:_V W) \subseteq (A:_V W) + Soc(V)$ or $ab \in (A:_V W) + Soc(V)$. Then $(A:_V W)$ is $Soc\text{-}T\text{-}ABSO$ ideal of V.*

Proposition 3.9

Let W be a cyclic an $V-module$ and A is $Soc\text{-}T\text{-}ABSO$ submodule of W. Then $(A + Soc(W):_V W)$ is $Soc\text{-}T\text{-}ABSO$ ideal of V.

Proof :Let $abc \in (A + Soc(W):_V W)$ for some $a,b,c \in V$ and let $W =< x >$ for some $x \in W$ since W is a cyclic. Now, assume that $ab \notin (A + Soc(W): W) + SocV$ and $bc \notin (A + Soc(W): W) + SocV$, then $ab \notin (A + Soc(W): W)$ and $bc \notin (A + Soc(W): W) \exists y,z \in V$ such that $ab(yx) \notin A + Soc(W)$ and $ab(zx) \notin A + Soc(W)$. Since $abc < x > \subseteq A + Soc(W)$, but $ab < x > \nsubseteq A + Soc(W)$ and $bc < x > \nsubseteq A + Soc(W)$, then $ac \in (A + Soc(W): W)$ since A is $Soc\text{-}T\text{-}ABSO$ submodule of W, then $ac \in (A + Soc(W): W) + Soc(V)$. Thus $(A + Soc(W): W)$ is $Soc\text{-}T\text{-}ABSO$ ideal of V.*

Proposition 3.10

If A is Soc-T-ABSO submodule of an $V - module$ W, with $soc(W) \subseteq A$ then $(A + Soc(W):_V < x >)$ is $Soc\text{-}T\text{-}ABSO$ ideal of V for all $x \in W - (A + Soc(W))$.

Proof :Let $x \in W - \left(A + Soc(W)\right)$, then $(A + Soc(W):_V < x >)$ is a proper ideal of V*

Let $abc \in (A + Soc(W):_V < x >)$ for some $a,b,c \in V$, hence $abc < x > \subseteq A + Soc(W)$. So that $abc < x > \subseteq A$, but A is $Soc\text{-}T\text{-}ABSO$ submodule of W, then $a < x > \subseteq A + Soc(W)$ or $bc < x > \subseteq A + Soc(W)$ or $abc \in (A + Soc(W): W)$

If $a < x > \subseteq A + Soc(W)$ and hence $a \in (A + Soc(W):_V < x >)$ so we get $a \in (A + Soc(W):_V < x >) + Soc(V)$ if $bc < x > \subseteq A + Soc(W)$, hence $bc \in (A + Soc(W):_V < x >)$, then $bc \in (A + Soc(W):_V < x >) + Soc(V)$. Thus $(A + Soc(W):_V < x >)$ is $Soc\text{-}T\text{-}ABSO$ ideal of V.

proposition 3.11

If A be a proper submodule of an $V - module$ W and $soc(W) \subseteq A$, then the following expressions are equivalent :

1. *A is $Soc\text{-}T\text{-}ABSO$ submodule of W*
2. *$(A + Soc(W):_W I)$ is $Soc\text{-}T\text{-}ABSO$ submodule of W for all ideal I of V with $IW \nsubseteq A + Soc(W)$.*
3. *$(A + Soc(W):_W < a >)$ is Soc-T-ABSO submodule of W for all a of V with $aW \nsubseteq A + Soc(W)$.*

Proof :$(1) \to (2)$ *let I be ideal of V with $IW \nsubseteq A + Soc(W)$. Then $(A + Soc(W):_W I)$ is a proper submodule of W, let $a,b \in V$, $x \in W$ such that $abx \in (A + Soc(W):_W I)$, hence $ab(Ix) \subseteq A + Soc(W)$, then $ab(Ix) \subseteq A$, but A is $Soc\text{-}T\text{-}ABSO$ submodule of W. Then either $a(Ix) \subseteq A + Soc(W)$ or $b(Ix) \subseteq A + Soc(W)$ or $ab \in (A + Soc(W):_V W) \subseteq \left((A + Soc(W):_W I):_V W\right)$ and hence $ax \in (A + Soc(W):_W I)$ or $bx \in (A + Soc(W:_W I)$ or $ab \in \left((A + Soc(W):_W I):_V W\right)$, then $(A + Soc(W):_W I)$ is $Soc\text{-}T\text{-}ABSO$ submodule of W*

$(2) \to (3)$ *it is clear*

$(3) \to (1)$ *if $a = 1$ then $(A + Soc(W):_W < 1 >) = A + Soc(W)$, then $A + Soc(W)$ is $Soc\text{-}T\text{-}ABSO$ submodule of W since $Soc(W) \subseteq A$. Thus A is $Soc\text{-}T\text{-}ABSO$ submodule of W.*

Proposition 3.12

Let A, B *and* D *be submodules of an* $V-module\ W$ *with* $D\subseteq B\subseteq Soc(A)$ *and if* B *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* A, *then* D *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* A.

Proof :*Let* $abx\in D$ *for* $a,b\in V$ *and* $x\in A$ *since* $D\subseteq B$, *then* $abx\in B$, *but B is Soc-T-ABSO submodule of* A. *This means that either* $ax\in B+Soc(A)$ *or* $bx\in B+Soc(A)$ *or* $ab\in(B+Soc(A):_V A)$, *then either* $ax\in Soc(A)\subseteq D+Soc(A)$ *or* $bx\in D+Soc(A)$ *or* $abA\subseteq Soc(A)\subseteq D+soc(A)$ *since* $B\subseteq Soc(A)$. *Thus* D *is a* $Soc\text{–}T\text{–}ABSO$ *submodule of* A

Proposition 3.13

Let A *be a proper submodule of an* $V-module\ W$ *and* $Soc(W)\subseteq A$. *Then* A *is* $T\text{–}ABSO$ *submodule of* W *if and only if* A *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* W.

Proof :$\Longrightarrow$) *it is clear*

$\Longleftarrow$) *let* $abx\in A$ *for some* $a,b\in V$, $x\in W$. *Since* A *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* W, *then* $ax\in A+Soc(W)$ *or* $bx\in A+Soc(W)$ *or* $ab\in(A+Soc(W):_V W)$, *but* $Soc(W)\subseteq A$, *then* $ax\in A$ *or* $bx\in A$ *or* $ab\in(A:_V W)$. *Thus* A *is* $T\text{–}ABSO$ *submodule of* W .

Theorem 3.14

Let W , W' *be two* $V\text{–}modules$, $f:W\to W'$ *be an isomorphism and* $Im(f)\leq_e W'$
If B *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* W' *then* $f^{-1}(B)$ *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* W.

Proof :*Since* $B<W'$, *then* $f^{-1}(B)<W$ *since* f *is epimorphisim .Let* $abx\in f^{-1}(B)$ *for some* $a,b\in V$, $x\in W$, *hence* $abf(x)\in B$. *But* B *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* W', *then either* $af(x)\in B+Soc(W')$ *or* $bf(x)\in B+Soc(W')$ *or* $ab\in(B+Soc(W'):W')$ *then* $ax\in f^{-1}(B)+Soc(W)$ *or* $bx\in f^{-1}(B)+Soc(W)$ *or* $ab\in(B+Soc(W'):W')$ *since* $f^{-1}\big(Soc(W')\big)=Soc(W), if$ $f:W\to W'$ *be an monomorphism and* $Im(f)\leq_e W'$ *by Proposition*(2.9)

If $ab\in(B+Soc(W'):W')$, *so that* $abW'\subseteq B+Soc(W')$, *but* $f(W)\subseteq W'$, *hence* $abf(W)\subseteq B+Soc(W')$ *and so that* $abW\subseteq f^{-1}(B)+Soc(W)$, *then* $ab\in(f^{-1}(B)+Soc(W):W)$. *Thus* $f^{-1}(B)$ *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* W

Theorem 3.15

Let W , W' *be two* $V\text{–}modules$, $f:W\to W'$ *be an epimorphism. If* H *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* W, *with* $\ker f\subseteq H$, *then* $f(H)$ *is Soc–T–ABSO submodule of* W'

Proof:*Let* $aby\in f(H)$, *where* $y\in W'$, $a,b\in V$ *and* $y=f(x)$ *for some* $x\in W$, *since f is epimorphism, then* $abf(x)\in f(H)$, *so that* $abf(x)=f(h)$ *for some* $h\in H$. *Then we have* $f(abx)-f(h)=0$, *hence* $abx-h\in\ker f\subseteq H$. *So that* $abx\in H$, *but* H *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* W. *Then either* $ax\in H+Soc(W)$ *or* $bx\in H+Soc(W)$ *or* $ab\in(H+Soc(W):_V W)$

If $ax\in H+Soc(W)$, *then* $f(ax)\in f\big(H+Soc(W)\big)$ *and hence* $af(x)\in f(H)+f\big(Soc(W)\big)$ *Thus* $ay\in f(H)+Soc(W')$ $\Big($*since* $f\big(Soc(W)\big)\subseteq Soc(W')\Big)$ *similarly if* $bx\in H+Soc(W)$, *hence* $by\in f(H)+Soc(W')$

Finally if $ab\in(H+Soc(W):W)$, *so that* $abW\subseteq H+Soc(W)$, *then* $f(abW)\subseteq f\big(H+Soc(W)\big)$, *hence* $abW'\subseteq f(H)+f\big(Soc(W)\big)$, *so* $abW'\subseteq f(H)+Soc(W')$.

Then $ab\in(f(H)+Soc(W'):_V W')$ *thus* $f(H)$ *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* W'

Corollary 3.16

Let W *be an* V *– module and* A *and* B *be submodules of* W *with* $B\subseteq A$ *then* A *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* W. *Then* $\frac{A}{B}$ *is* $Soc\text{–}T\text{–}ABSO$ *submodule of* $\frac{W}{B}$.

Proof : *consider that the canonical homomorphism* $\pi:W\to\frac{W}{B}$ *defined by* $\pi(x)=x+B\ \forall x\in W$
Then the proof is clear by Theorem(3.15)

Proposition 3.17

If A is $Soc\text{–}T\text{–}ABSO$ submodule of an $V-module\, W$, such that $Soc(W)\subseteq A$, then $(A:_V W)$ is $Soc\text{–}T\text{–}ABSO$ ideal of V.

Proof :*Let $abc\in(A:_V W)$ for some $a,b,c\in V$ then $abcW\subseteq A$, but A is $Soc\text{–}T\text{–}ABSO$ submodule of W, then $a(cW)\subseteq A+Soc(W)$ or $b(cW)\subseteq A+Soc(W)$ or $ab\in(A+Soc(W):_V W)$.* Since $Soc(W)\subseteq A$,we get $a(cW)\subseteq A$ or $b(cW)\subseteq A$ or $abW\subseteq A$, hence $ac\in(A:_R W)$ or $bc\in(A:_V W)$ or $ab\in(A:_V W)$, so that $ac\in(A:_V W)+Soc(V)$ or $bc\in(A:_V W)+Soc(V)$ or $ab\in(A:_V W)+Soc(V)$. Thus $(A:_V W)$ is $Soc\text{–}T\text{–}ABSO$ ideal of V.

Remark 3.18

In Proposition(3.17) if $Soc(W)\nsubseteq A$, then $(A:_V W)$ is not necessary $Soc\text{–}T\text{–}ABSO$ ideal of V, for example: in Z-module Z_{12}, $(\bar{0})$ is $Soc\text{–}T\text{–}ABSO$ submodule, but $\left((\bar{0}):_Z Z_{12}\right)=12Z$ is not $Soc\text{–}T\text{–}ABSO$ ideal of Z, since $2\cdot 2\cdot 3\in 12Z$, but $2\cdot 3\notin 12Z+Soc(Z)=12Z+(0)=12Z$ and $2\cdot 2\notin 12Z+Soc(Z)=12Z$.

Proposition 3.19

Let A,B be proper submodules of an $V-module\, W$. If A,B are $Soc\text{–}T\text{–}ABSO$ submodules of W with $A\subseteq Soc(W)$ and $B\subseteq Soc(W)$, then $A\cap B$ is $Soc\text{–}T\text{–}ABSO$ submodule of W.

Proof :*Let $abx\in A\cap B$ for some $a,b\in V$, $x\in W$. Then $abx\in A$ and $abx\in B$, but A and B are $Soc\text{–}T\text{–}ABSO$ submodules of W, then either $ax\in A+Soc(M)$ or $bx\in A+Soc(W)$ or $ab\in(A+Soc(W):_V W)$ and either $ax\in B+Soc(W)$ or $bx\in B+Soc(M)$ or $ab\in(B+Soc(W):_V W)$, but $A,B\subseteq Soc(W)$, hence $A\cap B\subseteq Soc(W)$. So that $ax\in Soc(W)$ or $bx\in Soc(W)$ or $ab\in(Soc(W):_V W)$. Then either $ax\in A\cap B+Soc(W)$ or $bx\in A\cap B+Soc(W)$ or $ab\in(A\cap B+Soc(W):_V W)$. Thus $A\cap B$ is $Soc\text{–}T\text{–}ABSO$ submodule of W.*

Proposition 3.20

Let A be a proper submodule of an $V-module\, W$, with $A=\cap_{i\in\Lambda}B_i$ where B_i is $T\text{–}ABSO$ submodule of W for each $i\in\Lambda$, then A is $Soc\text{–}T\text{–}ABSO$ submodule of W.

Proof : *Suppose that $abx\in A$ for some $a,b\in V, x\in W$, hence $abx\in\cap_{i\in\Lambda}B_i$, and so that $abx\in B_i,\forall i\in\Lambda$.Since B_i is $T\text{–}ABSO$ in W, then either $ax\in B_i$ or $bx\in B_i$ or $ab\subseteq(B_i:_V W)$,$\forall i\in\Lambda$, hence $ax\in\cap_{i\in\Lambda}B_i$ or $bx\in\cap_{i\in\Lambda}B_i$ or $ab\in(\cap_{i\in\Lambda}B_i:_V W)$, then $ax\in\cap_{i\in\Lambda}B_i+Soc(W)$ or $bx\in\cap_{i\in\Lambda}B_i+Soc(W)$ or $ab\in(\cap_{i\in\Lambda}B_i+Soc(W):_V W)$.*

Thus $A=\cap_{i\in\Lambda}B_i$ is $Soc\text{–}T\text{–}ABSO$ submodule of W.

Proposition 3.21

Let A be a proper submodule of $V-module\, W'$ and let W, W' be $V-modules$, with $A+Soc(W')$ is $T\text{–}ABSO$ submodule of W'. Then a proper submodule $Hom_V\big(W,A+soc(W')\big)$ of $Hom_V(W,W')$ is a $Soc\text{–}T\text{–}ABSO$ submodule of $Hom_V(W,W')$.

Proof :*Let $abf\in Hom_V\big(W,A+Soc(W')\big)$, for some $a,b\in V$, $f\in Hom_V(W,W')$. Then for all $x\in W$, $abf(x)\in A+Soc(W')$. But $A+Soc(W')$ is $T\text{–}ABSO$ submodule of W'. Then either $af(x)\in A+Soc(W')$ or $bf(x)\in A+Soc(W')$ of $abW'\subseteq A+Soc(W')$ and hence $af\in Hom_V\big(W,A+Soc(W')\big)+Soc\big(Hom_V(W,W')\big)$, since $\big[Hom_V\big(W,A+Soc(W')\big)\subseteq Hom_R\big(W,A+Soc(W')\big)+Soc\big(Hom_V(W,W')\big)\big]$*

or $bf\in Hom_V\big(W,A+Soc(W')\big)+Soc\big(Hom_V(W,W')\big)$ or $ab\,Hom_V(W,W')\subseteq Hom_V\big(W,A+Soc(W')\big)$ then $Hom_V\big(W,A+Soc(W')\big)$ is $Soc\text{–}T\text{–}ABSO$ submodule of $Hom_V(W,W')$.

Proposition 3.22

Let A be T-ABSO submodule of an $V-module\, W$, with $A\subseteq Soc(W)$, then $(A:_W yz)=W$ or $(A:_W yz)\subseteq(Soc(W):_W y)\cup(Soc(W):_W z)$ $for\ all\ y,z\in V$.

Proof:

Let A be $T\text{–}ABSO$ submodule of W if $yz\in(A:_V W)$, then $(A:_W yz)=W$ $for\ all\ y,z\in V$

if $yz \notin (A:_V W)$*, then let* $x \in (A:_W yz)$*, hence* $yzx \in A$*, but* A *is* $T\!-ABSO$ *submodule, then either* $yx \in A$ *or* $zx \in A$ *, and so* $yx \in A + Soc(W)$ *or* $zx \in A + Soc(W)$*, but* $A \subseteq Soc(W)$*, then* $yx \in Soc(W)$ *or* $zx \in Soc(W)$*, hence* $x \in (Soc(W):_W y)$ *or* $x \in (Soc(W):_W z)$*.*
So, we get $(A:_W yz) \subseteq (Soc(W):_W y) \cup (Soc(W):_W z)\ for\ all\ y, z \in V$*.*

Proposition 3.23

Let A *be a proper submodule of an* $V\!-module\ W$*, and* $(A + Soc(W):_W yz) = W$ *or* $(A + Soc(W):_W yz) \subseteq (Soc(W):_W y) \cup (Soc(W):_W z)$ $for\ all\ y, z \in V$ *then* A *is* $Soc\!-T\!-ABSO$ *submodule of* W*.*

Proof: *Let* $yzx \in A$ *for some* $y, z \in V, x \in W$*, suppose that* $yz \notin (A + Soc(W):_V W)$*, then* $(A + Soc(W):_W yz) \neq W$*. Now, let* $x \in (A + Soc(W):_W yz) \subseteq (Soc(W):_W y)$ *or* $x \in (A + Soc(W):_W yz) \subseteq (Soc(W):_W z)$*, then* $yx \in Soc(W)$ *or* $zx \in Soc(W)$*, hence* $yx \in A + Soc(W)$ *or* $zx \in A + Soc(W)$*. Thus* A *is* $Soc\!-T\!-ABSO$ *submodule of* W*.*

Proposition 3.24

Let A *be a proper submodule of an* $V - module\ W$*. If* A *is intersection complement and addition complement of* $Soc(W)$*, then* A *is* $Soc\!-T\!-ABSO$ *submodule of* W*.*

Proof :*Let* $abx \in A$ *for some* $a, b \in V$ *,* $x \in W$*. Since* A *is intersection complement and addition complement of* $Soc(W)$*, then* $A \oplus Soc(W) = W$*, hence* $A + Soc(W) = W$ *and* $A \cap Soc(W) = \{0\}$*, implies that* $ax \in A + Soc(W)$ *and* $bx \in A + Soc(W)$*. Hence* A *is* $Soc\!-T\!-ABSO$ *submodule of* W*.*

Proposition 3.25

Let A *be a proper submodule of an* $V - module\ W$*. If* A *is addition complement of* $Soc(W)$*, then* A *is* $Soc\!-T\!-ABSO$ *submodule of* W*.*

Proof: *Let* $abx \in A$ *for some* $a, b \in V$ *,* $x \in W$ *Since* A *is addition complement of* $Soc(W)$*, then* $A + Soc(W) = W$ *and hence* $ax \in A + Soc(W)$ *and* $bx \in A + Soc(W)$*. Hence* A *is* $Soc\!-T\!-ABSO$ *submodule of* W*.*

Corollary 3.26

If A adco of B in W *then* $(A + B) + Soc(W)$ *is T-ABSO submodule of* W*.*

Proposition 3.27

Let A *and* B *be submodules of an* $V - module\ W$*. If* $\langle A \cup B \rangle$ *is* $T\!-ABSO$ *submodule of* W *and* $B \subseteq Soc(W)$*. Then* A *is* $Soc\!-T\!-ABSO$ *submodule of* M

Proof:*Let* $abx \in A$ *for some* $a, b \in R$ *,* $x \in W$*, hence* $abx \in \langle A \cup B \rangle$*, then either* $ax \in \langle A \cup B \rangle$ *or* $bx \in \langle A \cup B \rangle$ *or* $abW \subseteq \langle A \cup B \rangle$*. Since* $\langle A \cup B \rangle = A + B$ *then either* $ax \in A + B$*or* $bx \in A + B$ *or* $abW \subseteq A + B$*, but* $B \subseteq Soc(W)$*. hence either* $ax \in A + Soc(W)$ *or* $bx \in A + Soc(W)$ *or* $abW \subseteq A + Soc(W)$*, so that* A *is* $Soc\!-T\!-ABSO$ *submodule of* W*.*

4 CONCLUSION

As a new generalization of the T-ABSO submodule, the Soc-T-ABSO submodule are introduced. The following are the study's main findings:

(1) It is evident that every $T\!-ABSO$ *submodule is* $Soc\!-T\!-ABSO$ *submodule, but the converse incorrect in general see Remark and Example(3.3).*
(2) It is evident that every quasi-$prime$ *submodule is* $Soc\!-T\!-ABSO$ *submodule, but the converse incorrect in general see Remark and Example(3.3).*

(3) *A proper submodule A of an $V-module$ W is $Soc\text{–}T\text{–}ABSO$ submodule of W if and only if $IJH \subseteq A$ for some ideals I,J of V and some submodule H of W then either $IH \subseteq A+Soc(W)$* or $JH \subseteq A+Soc(W)$ or $IJ \subseteq (A+Soc(W):_V W)$.

(4) *If a proper submodule A of an $V-module$ W is $T\text{–}ABSO$ submodule of W, then $(A:_V W)$ is $Soc\text{–}T\text{–}ABSO$ ideal of V.*

(5) *If W be a cyclic an $V-module$ and A is $Soc\text{–}T\text{–}ABSO$ submodule of W. Then $(A+Soc(W):_V W)$ is $Soc\text{–}T\text{–}ABSO$ ideal of V.*

(6) *If A,B are $Soc\text{–}T\text{–}ABSO$ submodules of W with $A \subseteq Soc(W)$ and $B \subseteq Soc(W)$, then $A \cap B$ is $Soc\text{–}T\text{–}ABSO$ submodule of W.*

(7) *When A be a proper submodule of $V-module$ W' such that W,W' be $V-modules$, with $A+Soc(W')$ is $T\text{–}ABSO$ of W'. Then a proper submodule $Hom_R\big(W,A+soc(W')\big)$ of $Hom_V(W,W')$ is a $Soc\text{–}T\text{–}ABSO$ submodule of $Hom_V(W,W')$.*

(8) *If A is intersection complement and addition complement of $Soc(W)$, then A is $Soc\text{–}T\text{–}ABSO$ submodule of W.*

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