A New Type of Metric space Via Proximit Structure

Dhuha Abdulameer Kadhim
University of Kufa, Faculty of Education for Girls , Department of Mathematics
Najaf, Iraq
dhuhaa.ebada@uokufa.edu.iq
Boushra Y. Hussein
University of Al-Qadisiyah, College of Education, Department of Mathematics
Al-Qadisiyah, Iraq
boushra.alshebani@qu.edu.iq

ABSTRACT

The relationship between Dh-functional and the tower, as well as certain fundamental facts and examples, are used to investigate the idea of a new type of proximit space. Theorems relating to proximit space should also be included. Additionally, we demonstrate how proximit and metric space are connected.

KEYWORDS: proximit space, metric space, Dh-functional, tower, Lo-tower

1 INTRODUCTION

Lowen [8] showed how approach theory is frequently hidden behind other theories, and how getting it to the foreground and implementing it consistently enhances the theory, and he explained how contractions, continuous functions, compact sets, and convergent sequences are commonly dealt with in topology and analysis, as in many mathematical theories. Lowen [12] has both a completion and a compactification construction in an appropriate subcategory of approach. as well as the notion of contraction by an approach system. R. Lowen and K. Robeys [5,6] established some basic definitions and theorems in approach space which is a topological and metric space extension. More information about approach space can be found in [4]. M. Sion and W. Haute [14] investigated an alternative notion of approach spaces based on approach cores. [15] presented the relationship between approach space and metric space which is called a sober metric approach space. In [3,13] considered distances. Functional analysis applications of approach theory are discussed in [7,9,10,11]. A new kind of approach space made by [2]. Furthermore, Abbas and Hussein worked the completion normed approach space in [1]. This research looks into the regularity of functionals between sets. In section 2, using the concepts of Dh-functional and tower, the definitions of Dh-functional and proximit space are presented. The examples of this idea is studied. In section 3, we receive the most important theorems concerning this object.

2 PRELIMINARIES

All of the subsets of the set $\widetilde{\mathbb{X}}$ is symbolized by $2^{\widetilde{\mathbb{X}}}$.

Definition 2.1:[6]

If a function $\delta: \widetilde{\mathbb{X}} \times 2^{\widetilde{\mathbb{X}}} \to [0,\infty]$ mollifies the following conditions, it is said to be a distance on $\widetilde{\mathbb{X}}$ for all $\tilde{a} \in \widetilde{\mathbb{X}}$ and $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}$

(D1) $\delta(\tilde{a}, \{\tilde{a}\}) = 0$,

(D2) $\delta(\tilde{a}, \emptyset) = \infty$,

(D3) $\delta(\hat{a}, \hat{\mathcal{A}} \cup \hat{\mathcal{B}}) = \min\left(\delta(\hat{a}, \hat{\mathcal{A}}), \delta(\hat{a}, \hat{\mathcal{B}})\right)$,

(D4) $\forall \varepsilon \in [0,\infty]$: $\delta(\hat{a}, \hat{\mathcal{A}}) \le \delta(\hat{a}, \hat{\mathcal{A}}^{(\varepsilon)}) + \varepsilon$ where $\hat{\mathcal{A}}^{(\varepsilon)} = \{\hat{a} \in \widetilde{\mathbb{X}} : \delta(\hat{a}, \hat{\mathcal{A}}) \le \varepsilon\}$.

Definition 2.2:[6]

A collection of functions $t_\varepsilon: 2^{\widetilde{\mathbb{X}}} \to 2^{\widetilde{\mathbb{X}}}$, $\varepsilon \in \mathfrak{R}^+$ is said to be a tower on $\widetilde{\mathbb{X}}$ if the following conditions are fulfilled for every $\varepsilon, \gamma \in \mathfrak{R}^+$:

(To1) $\forall \hat{\mathcal{A}} \in 2^{\widetilde{\mathbb{X}}}$: $\hat{\mathcal{A}} \subset t_\varepsilon(\hat{\mathcal{A}})$,

(To2) $t_\varepsilon(\emptyset) = \emptyset$

(To3) $\forall \hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\mathbb{X}}$: $t_\varepsilon(\hat{\mathcal{A}} \cup \hat{\mathcal{B}}) = t_\varepsilon(\hat{\mathcal{A}}) \cup t_\varepsilon(\hat{\mathcal{B}})$, (To4) $\forall \hat{\mathcal{A}} \in 2^{\widetilde{\mathbb{X}}} : t_\varepsilon(t_\gamma(\hat{\mathcal{A}})) \subset t_{\varepsilon+\gamma}(\hat{\mathcal{A}})$,

(To5) $\forall \hat{\mathcal{A}} \in 2^{\widetilde{\mathbb{X}}} : t_\varepsilon(\hat{\mathcal{A}}) = \bigcap_{\varepsilon<\gamma} t_\gamma(\hat{\mathcal{A}})$,

Take note that with (To3) and (To5), we get

$$\forall \hat{\mathcal{A}} \subset \hat{\mathcal{B}} \subset \widetilde{\mathbb{X}}, \forall \alpha \le \beta \in \mathfrak{R}^+ : t_\alpha(\hat{\mathcal{A}}) \subset t_\beta(\hat{\mathcal{B}}).$$

Definition 2.3:[6]

An operator $cl: 2^{\widetilde{\mathbb{X}}} \to 2^{\widetilde{\mathbb{X}}}$ that determines the qualities of a pretopology on a set $\widetilde{\mathbb{X}}$ for all $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}$ is determined by an operator that determines the properties of an operator on a set.

1. $\hat{\mathcal{A}} \subset cl(\hat{\mathcal{A}})$
2. $cl(\emptyset) = \emptyset$,
3. $cl(\hat{\mathcal{A}} \cup \hat{\mathcal{B}}) = cl(\hat{\mathcal{A}}) \cup cl(\hat{\mathcal{B}})$.

This operator is said to be a pretopological closure operator. We call a set $\widetilde{\mathbb{X}}$ with a pretopology a pretopological space**.**

Proposition 2.4:[6]

1. If $(t_\varepsilon)_{\varepsilon\in\mathfrak{R}^+}$ is tower, then $\forall \varepsilon \in \mathfrak{R}^+, t_\varepsilon$ is a pretopological closure operator, and (To4) and (To5) are true.
2. If t_ε is a pretopological closure operator, and (To4) and (To5) are fulfilled, then t_0 is a is a topological closure operator, for each $\varepsilon > 0$, t_ε is a pretopological closure operator, and (To4) and (To5) are true.
3. If t_0 is a is a topological closure operator, for all $\varepsilon > 0$, t_ε is a pretopological closure operator, and (To4) and (To5) are fulfilled, then $(t_\varepsilon)_{\varepsilon\in\mathfrak{R}^+}$ is tower.

3 Proximit Metric Space

The relation between metric and proximit space, Dh-functional, and Lo-tower are presented in this section.

Definition 3.1

A functional $\rho: 2^{\widetilde{\mathbb{X}}} \times 2^{\widetilde{\mathbb{X}}} \to [0,\infty]$ is said to be an Dh-functional on $\widetilde{\mathbb{X}}$ ($\widetilde{\mathbb{X}}$ is a set) if it satisfies the following:

(h1) $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \rho(\hat{\mathcal{B}}, \hat{\mathcal{A}}) \quad \forall \hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}$

(h2) $\hat{\mathcal{A}} = \emptyset \vee \hat{\mathcal{B}} = \emptyset$ *then* $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \infty \quad \forall \hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}$

(h3) $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = 0$ *then* $\hat{\mathcal{A}} \cap \hat{\mathcal{B}} \neq \emptyset \quad \forall \hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}$

(h4) $\rho(\mathcal{A}, \hat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \min\{\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}), \rho(\hat{\mathcal{A}}, \hat{\mathcal{C}})\} \quad \forall \hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{\widetilde{\mathbb{X}}}$

(h5) $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \rho(\hat{\mathcal{A}}^\varepsilon, \hat{\mathcal{B}}^\eta) + \varepsilon + \eta \quad \forall \hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}, \forall \varepsilon, \eta \in [0,\infty]$

For every $\hat{\mathcal{A}} \in 2^{\widetilde{\mathbb{X}}}, \varepsilon \in [0,\infty]$, we write $t_\varepsilon(\hat{\mathcal{A}}) := \{\tilde{a} \in \widetilde{\mathbb{X}} \mid \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \le \varepsilon\}$. Therefore, the triple $(\widetilde{\mathbb{X}}, \rho, t_\varepsilon)$ is called proximit space.

Example 3.2:

Let $\widetilde{\mathbb{X}} = [0,\infty]$, define $\rho: 2^{[0,\infty]} \times 2^{[0,\infty]} \to [0,\infty]$ by

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \begin{cases} 0 & \hat{\mathcal{A}}, \hat{\mathcal{B}} \text{ unbounded} \\ \infty & \hat{\mathcal{A}}, \hat{\mathcal{B}} \text{ bounded} \\ \inf\limits_{\ddot{n}\in\hat{\mathcal{A}}} \inf\limits_{\ddot{m}\in\hat{\mathcal{B}}} |\ddot{n} - \ddot{m}| & \ddot{n}, \ddot{m} < \infty \end{cases}$$

Then $(\widetilde{\mathbb{X}}, \rho, t_\varepsilon)$ is proximit space.

Solve:

Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{[0,\infty]}$

(h1) If $\ddot{n}, \ddot{m} < \infty$

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \inf_{\ddot{n}\in\hat{\mathcal{A}}} \inf_{\ddot{m}\in\hat{\mathcal{B}}} |\ddot{n} - \ddot{m}| = \inf_{\ddot{m}\in\hat{\mathcal{A}}} \inf_{\ddot{n}\in\hat{\mathcal{B}}} |\ddot{m} - \ddot{n}| = \rho(\hat{\mathcal{B}}, \hat{\mathcal{A}})$$

If $\hat{\mathcal{A}}, \hat{\mathcal{B}}$ *unbounded*

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = 0 = \rho(\hat{\mathcal{B}}, \hat{\mathcal{A}})$$

If $\hat{\mathcal{A}}, \hat{\mathcal{B}}$ *bounded*

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \infty = \rho(\hat{\mathcal{B}}, \hat{\mathcal{A}}).$$

(h2) If $\hat{\mathcal{A}}, \widehat{\mathcal{B}}$ *unbounded*

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} |\ddot{n}, \dot{m}| = \infty$$

If $\hat{\mathcal{A}}, \widehat{\mathcal{B}}$ *bounded*, then $\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = 0$

(h3) Let $\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = 0$, so $\inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} |\ddot{n} - \dot{m}| = 0$. Therefore $\ddot{n} = \dot{m}$, thus $\ddot{n} \in \hat{\mathcal{A}} \cap \widehat{\mathcal{B}}$. Then $\hat{\mathcal{A}} \cap \widehat{\mathcal{B}} \neq \emptyset$.

(h4) If $\hat{\mathcal{A}}, \widehat{\mathcal{B}}$ *bounded*

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}} \cup \hat{\mathcal{C}}} |\ddot{n} - \dot{m}| = 0 = \min\{\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}), \rho(\hat{\mathcal{A}}, \hat{\mathcal{C}})\}$$

If $\ddot{n}, \dot{m} < \infty$,

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \inf_{\ddot{n} \in \mathcal{A}} \inf_{\dot{m} \in \widehat{\mathcal{B}} \cup \hat{\mathcal{C}}} |n - m| = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \left(\min\left\{ \inf_{\dot{m} \in \widehat{\mathcal{B}}} |\ddot{n} - \dot{m}|, \inf_{\dot{m} \in \hat{\mathcal{C}}} |\ddot{n} - \dot{m}| \right\}\right)$$
$$= \min\left(\inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} |\ddot{n}, \dot{m}|, \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \hat{\mathcal{C}}} |\ddot{n} - \dot{m}| \right) = \min(\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}), \rho(\hat{\mathcal{A}}, \hat{\mathcal{C}}))$$

If $\hat{\mathcal{A}}, \widehat{\mathcal{B}}$ *unbounded*

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \infty = \min\{\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}), \rho(\hat{\mathcal{A}}, \hat{\mathcal{C}})\}$$

(h5) If $\ddot{n}, \dot{m} < \infty$, $\hat{\mathcal{A}}, \widehat{\mathcal{B}} \in 2^{[0,\infty]}$, $\varepsilon, \eta \in [0, \infty]$

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} |n - m| \leq \inf_{\ddot{n} \in \hat{\mathcal{A}}^{\varepsilon}} \inf_{\dot{m} \in \widehat{\mathcal{B}}^{\varepsilon}} |\ddot{n} - \dot{m}| + \varepsilon + \eta = \rho(\hat{\mathcal{A}}^{\varepsilon}, \widehat{\mathcal{B}}^{\eta}) + \varepsilon + \eta$$

If $\hat{\mathcal{A}}, \widehat{\mathcal{B}}$ *unbounded*, $\varepsilon, \eta \in [0, \infty]$

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} |\ddot{n} - \dot{m}| = 0 \leq \rho(\hat{\mathcal{A}}^{\varepsilon}, \widehat{\mathcal{B}}^{\eta}) + \varepsilon + \eta$$

If $\hat{\mathcal{A}}, \widehat{\mathcal{B}}$ *bounded*, $\varepsilon, \eta \in [0, \infty]$

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} |\ddot{n} - \dot{m}| = \infty \leq \rho(\hat{\mathcal{A}}^{\varepsilon}, \widehat{\mathcal{B}}^{\eta}) + \varepsilon + \eta.$$

Example 3.3:

Let $\mathfrak{R}$ be a set of real numbers, then $(\mathfrak{R}^n, \rho, t_\varepsilon)$ is a proximit space.

Solve:

Let $\rho: 2^{\mathfrak{R}^n} \times 2^{\mathfrak{R}^n} \to [0, \infty]$ be a function defined by

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \begin{cases} \infty & \hat{\mathcal{A}} = \emptyset \text{ or } \widehat{\mathcal{B}} = \emptyset \\ \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} d(\ddot{n}, \dot{m}) & \hat{\mathcal{A}} \neq \emptyset \text{ and } \widehat{\mathcal{B}} \neq \emptyset \end{cases}$$

Let $\hat{\mathcal{A}}, \widehat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{\mathfrak{R}^n}$ such that $\ddot{n} = (n_1, n_2, \dots, n_n), \dot{m} = (m_1, m_2, \dots, m_n) \in \mathfrak{R}^n$,

(h1) if $\hat{\mathcal{A}} \neq \emptyset, \widehat{\mathcal{B}} \neq \emptyset$

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} d(\ddot{n}, \dot{m}) = \inf_{\dot{m} \in \widehat{\mathcal{B}}} \inf_{\ddot{n} \in \hat{\mathcal{A}}} d(\dot{m}, \ddot{n}) = \rho(\widehat{\mathcal{B}}, \hat{\mathcal{A}})$$

(h2) If $\hat{\mathcal{A}} = \emptyset$,

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \inf_{\emptyset \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} d(\emptyset, \dot{m}) = \infty$$

If $\widehat{\mathcal{B}} = \emptyset$

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\emptyset \in \widehat{\mathcal{B}}} d(\ddot{n}, \emptyset) = \infty$$

(h3) If $\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = 0$, so $\inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} d(\ddot{n}, \dot{m}) = 0$. Then $\ddot{n} = \dot{m} \Longrightarrow \ddot{n} \in \hat{\mathcal{A}} \cap \widehat{\mathcal{B}}$. Hence $\hat{\mathcal{A}} \cap \widehat{\mathcal{B}} \neq \emptyset$.

(h4) $$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}} \cup \hat{\mathcal{C}}} d(\ddot{n}, \dot{m}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \left(\min\left\{ \inf_{\dot{m} \in \widehat{\mathcal{B}}} d(\ddot{n}, \dot{m}), \inf_{\dot{m} \in \hat{\mathcal{C}}} d(\ddot{n}, \dot{m}) \right\}\right)$$
$$= \min\left\{ \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} d(n, m), \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \hat{\mathcal{C}}} d(n, m) \right\} = \min\{\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}), \rho(\hat{\mathcal{A}}, \hat{\mathcal{C}})\}$$

(h5) Let $\varepsilon, \eta \in [0, \infty]$

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}} \inf_{\dot{m} \in \widehat{\mathcal{B}}} d(\ddot{n}, \dot{m}) \leq \rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \inf_{\ddot{n} \in \hat{\mathcal{A}}^{\varepsilon}} \inf_{\dot{m} \in \widehat{\mathcal{B}}^{\eta}} d(\ddot{n}, \dot{m}) + \varepsilon + \eta$$
$$= \rho(\hat{\mathcal{A}}^{\varepsilon}, \widehat{\mathcal{B}}^{\eta}) + \varepsilon + \eta.$$

Proposition 3.4:

(1) Any (and all) of the following structures determine the finest (discrete) proximit space in a set $\widetilde{\mathbb{X}}$:

1. Dh-functional: $\rho: 2^{\widetilde{\mathbb{X}}} \times 2^{\widetilde{\mathbb{X}}} \to [0, \infty]$ where $\forall \tilde{a} \in \widetilde{\mathbb{X}}$ and $\hat{\mathcal{A}}, \widehat{\mathcal{B}} \subset \widetilde{\mathbb{X}}$,

$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \begin{cases} 0 & \tilde{a} \in \hat{\mathcal{A}} \cap \widehat{\mathcal{B}} \\ \infty & \tilde{a} \notin \hat{\mathcal{A}} \text{ or } \tilde{a} \notin \widehat{\mathcal{B}} \end{cases}$$

2. Tower: $(t_\varepsilon)_{\varepsilon\in\mathfrak{R}^+}$ where $\forall\varepsilon\in\mathfrak{R}^+$ and $\hat{\mathcal{A}},\hat{\mathcal{B}}\subset\tilde{\mathbb{X}}$, $t_\varepsilon(\hat{\mathcal{A}})=\hat{\mathcal{A}}, t_\varepsilon(\hat{\mathcal{B}})=\hat{\mathcal{B}}$.

(2) Using any (or all) of the following structures, determine the coarsest (indiscrete or trivial) proximit space on a set $\tilde{\mathbb{X}}$:

1. Dh-functional: $\rho: 2^{\tilde{\mathbb{X}}}\times 2^{\tilde{\mathbb{X}}}\to[0,\infty]$ such that $\forall\hat{\mathcal{A}},\hat{\mathcal{B}}\subset\tilde{\mathbb{X}}$,

$$\rho(\hat{\mathcal{A}},\hat{\mathcal{B}})=\begin{cases}0 & \hat{\mathcal{A}}\neq\emptyset \; and \; \hat{\mathcal{B}}\neq\emptyset\\ \infty & \hat{\mathcal{A}}=\emptyset \; or \; \hat{\mathcal{B}}=\emptyset\end{cases}$$

2. Tower: $(t_\varepsilon)_{\varepsilon\in\mathfrak{R}^+}$ where $\forall\varepsilon\in\mathfrak{R}^+$ and $\forall\hat{\mathcal{A}}\subset\tilde{\mathbb{X}}$,

$$t_\varepsilon(\hat{\mathcal{A}})=\begin{cases}\tilde{\mathbb{X}} & \hat{\mathcal{A}}\neq\emptyset\\ \emptyset & \hat{\mathcal{A}}=\emptyset\end{cases}$$

Proposition 3.5:

Every metric space is Proximit Space**.**

Proof

Given a metric space $(\tilde{\mathbb{X}}, d)$, define $\rho_d: 2^{\tilde{\mathbb{X}}}\times 2^{\tilde{\mathbb{X}}}\to[0,\infty]$ by

$$\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}})=\begin{cases}\infty, & \hat{\mathcal{A}}=\emptyset \; or \; \hat{\mathcal{B}}=\emptyset\\ \inf\limits_{\ddot{n}\in\hat{\mathcal{A}}}\inf\limits_{\dot{m}\in\hat{\mathcal{B}}} d(\ddot{n},\dot{m}), & \hat{\mathcal{A}}\neq\emptyset \; and \; \hat{\mathcal{B}}\neq\emptyset\end{cases}$$

Then ρ_d is distance on $\tilde{\mathbb{X}}$ and $(\tilde{\mathbb{X}},\rho_d)$ is said to be proximit distance generated by d or proximit metric space. Let $\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{\mathcal{C}}\in 2^{\tilde{\mathbb{X}}}$

1. If $\hat{\mathcal{A}}\neq\emptyset \; and \; \hat{\mathcal{B}}\neq\emptyset$

$$\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}})=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}} d(\ddot{n},\dot{m})=\inf_{\dot{m}\in\hat{\mathcal{B}}}\inf_{\ddot{n}\in\hat{\mathcal{A}}} d(\dot{m},\ddot{n})=\rho_d(\hat{\mathcal{B}},\hat{\mathcal{A}}).$$

2. If $\hat{\mathcal{A}}=\emptyset$, $\rho_d(\emptyset,\hat{\mathcal{B}})=\infty$.
   If $\hat{\mathcal{B}}=\emptyset$, $\rho_d(\hat{\mathcal{A}},\emptyset)=\infty$.
3. If $\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}})=0$, so $\inf\limits_{\ddot{n}\in\hat{\mathcal{A}}}\inf\limits_{\dot{m}\in\hat{\mathcal{B}}} d(\ddot{n},\dot{m})=0$. Then $\ddot{n}=\dot{m}\Longrightarrow\ddot{n}\in\hat{\mathcal{A}}\cap\hat{\mathcal{B}}$. Hence $\hat{\mathcal{A}}\cap\hat{\mathcal{B}}\neq\emptyset$.
4. If $\hat{\mathcal{A}}\neq\emptyset,\hat{\mathcal{B}}\neq\emptyset$. Then

$$\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}}\cup\hat{\mathcal{C}})=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}\cup\hat{\mathcal{C}}} d(\ddot{n},\dot{m})=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\left(\min\left\{\inf_{\dot{m}\in\hat{\mathcal{B}}} d(\ddot{n},\dot{m}),\inf_{\dot{m}\in\hat{\mathcal{C}}} d(\ddot{n},\dot{m})\right\}\right)$$
$$=\min\left(\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}} d(\ddot{n},\dot{m}),\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{C}}} d(\ddot{n},\dot{m})\right)=$$
$$\min\{\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}}),\rho_d(\hat{\mathcal{A}},\hat{\mathcal{C}})\}.$$

If $\hat{\mathcal{A}}=\emptyset$. Then

$$\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}}\cup\hat{\mathcal{C}})=\infty=\min\{\infty,\infty\}=\min\{\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}}),\rho_d(\hat{\mathcal{A}},\hat{\mathcal{C}})\}.$$

If $\hat{\mathcal{B}}=\emptyset$. Then

$$\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}}\cup\mathcal{C})=\infty=\min\{\infty,\infty\}=\min\{\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}}),\rho_d(\hat{\mathcal{A}},\mathcal{C})\}.$$

5. Let $\varepsilon,\eta\in[0,\infty]$, $\hat{\mathcal{A}},\hat{\mathcal{B}}\in 2^{\tilde{\mathbb{X}}}$. If $\mathcal{A}\neq\emptyset,\hat{\mathcal{B}}\neq\emptyset$

$$\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}})=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}} d(n,m)\leq\inf_{\ddot{n}\in\hat{\mathcal{A}}^\varepsilon}\inf_{\dot{m}\in\hat{\mathcal{B}}^\eta} d(\ddot{n},\dot{m})+\varepsilon+\eta=\rho_d(\hat{\mathcal{A}}^\varepsilon,\hat{\mathcal{B}}^\eta)+\varepsilon+\eta.$$

If $\hat{\mathcal{A}}=\emptyset$, then $\rho_d(\hat{\mathcal{A}},\hat{\mathcal{B}})=\infty$ and

$$\rho_d(\hat{\mathcal{A}}^\varepsilon,\hat{\mathcal{B}}^\eta)+\varepsilon+\eta=\inf_{\ddot{n}\in\hat{\mathcal{A}}^\varepsilon}\inf_{\dot{m}\in\hat{\mathcal{B}}^\eta} d(\ddot{n},\dot{m})+\varepsilon+\eta$$
$$\leq\inf_{\ddot{n}\in\hat{\mathcal{A}}^\varepsilon}\inf_{\dot{m}\in\hat{\mathcal{B}}^\eta} d(\ddot{n},\dot{m})+\varepsilon^n+\eta^n=\infty.$$

If $\hat{\mathcal{B}}=\emptyset$, we get

$$\rho_d(\hat{\mathcal{A}}^\varepsilon,\hat{\mathcal{B}}^\eta)+\varepsilon+\eta=\inf_{\ddot{n}\in\hat{\mathcal{A}}^\varepsilon}\inf_{\dot{m}\in\hat{\mathcal{B}}^\eta} d(\ddot{n},\dot{m})+\varepsilon+\eta$$
$$\leq\inf_{\ddot{n}\in\hat{\mathcal{A}}^\varepsilon}\inf_{\dot{m}\in\hat{\mathcal{B}}^\eta} d(\ddot{n},\dot{m})+\varepsilon^n+\eta^n=\infty$$

Theorem 3.6

If $(t_\varepsilon):\varepsilon\in\mathfrak{R}^+$ is a tower on $\hat{\mathbb{X}}$, then

$$\rho(\hat{\mathcal{A}},\hat{\mathcal{B}}):=\min\{\inf\{\varepsilon\in\mathfrak{R}^+|\hat{\mathcal{A}}\subseteq t_\varepsilon(\hat{\mathcal{B}})\},\inf\{\gamma\in\mathfrak{R}^+|\hat{\mathcal{B}}\subseteq t_\gamma(\mathcal{A})\}\}$$

$for\ all\ \hat{\mathcal{A}},\hat{\mathcal{B}}\in 2^{\tilde{\mathbb{X}}}$ is an Dh-functional on $\hat{\mathbb{X}}$.

Proof

Let $\hat{\mathcal{A}},\hat{\mathcal{B}}\in 2^{\tilde{\mathbb{X}}}$

(h1) $$\rho(\hat{\mathcal{A}},\hat{\mathcal{B}})=\min(\inf\{\varepsilon\in\mathfrak{R}^+|\ \hat{\mathcal{A}}\subseteq t_\varepsilon(\hat{\mathcal{B}})\},\ \inf\{\gamma\in\mathfrak{R}^+|\hat{\mathcal{B}}\subseteq t_\gamma(\hat{\mathcal{A}})\})$$

$$= \min(\inf\{\gamma \in \Re^+ |\ \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}, \inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}})\})$$
$$= \rho(\hat{\mathcal{B}}, \hat{\mathcal{A}})$$

(h2) if $\hat{\mathcal{A}} = \emptyset$, we have

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \min(\inf\{\varepsilon \in \Re^+ | \emptyset \subseteq t_\varepsilon(\hat{\mathcal{B}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\emptyset)\})$$
$$= \min(\inf\{\gamma \in \Re^+ | \emptyset \subseteq t_\gamma(\hat{\mathcal{B}})\}, \inf\{\varepsilon \in \Re^+ | \hat{\mathcal{B}} \subseteq \emptyset\}) = \infty$$

If $\hat{\mathcal{B}} = \emptyset$, we have

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \min(\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\emptyset)\}, \inf\{\gamma \in \Re^+ | \emptyset \subseteq t_\gamma(\hat{\mathcal{A}})\})$$
$$= \min(\inf\{\gamma \in \Re^+ | \hat{\mathcal{A}} \subseteq \emptyset\}, \inf\{\varepsilon \in \Re^+ | \emptyset \subseteq t_\gamma(\hat{\mathcal{A}})\}) = \infty.$$

(h3) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\mathbb{X}}$, if $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = 0$. We get

$$\min(\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}})\}, \inf\{\gamma \in \Re^+ |\ \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}) = 0,$$

then

$$(\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}})\}, \inf\{\gamma \in \Re^+ |\ \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}) = 0.$$

Thus

$$\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}})\} = 0\ \ \&\ \ \inf\{\gamma \in \Re^+ |\ \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\} = 0.$$

So that $\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}} \subseteq t_\varepsilon(\hat{\mathcal{B}}) = 0$ & $\hat{\mathcal{B}} \subseteq \hat{\mathcal{A}} \subseteq t_\gamma(\hat{\mathcal{A}}) = 0$. Take $\gamma, \varepsilon = 0$, then $\hat{\mathcal{A}} \subseteq t_0(\hat{\mathcal{B}}) \subseteq \hat{\mathcal{B}}$ and $\hat{\mathcal{B}} \subseteq t_0(\hat{\mathcal{A}}) \subseteq \hat{\mathcal{A}}$. Then $\hat{\mathcal{A}} \cap \hat{\mathcal{B}} \neq \emptyset$.

(h4) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{\tilde{\mathbb{X}}}$,

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}} \cup \hat{\mathcal{C}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}\}$$
$$= \min\{\inf\{\varepsilon \in \Re^+ |\ \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}}) \cup t_\varepsilon(\hat{\mathcal{C}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}\}$$
$$= \min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}}) \cup t_\varepsilon(\hat{\mathcal{C}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}\}$$
$$= \min\{\min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{A}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}, \inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{C}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}\}$$
$$= \min\{\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}), \rho(\hat{\mathcal{A}}, \hat{\mathcal{C}})\}$$

(h5) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}$, let $\theta, \vartheta \in [0, \infty]$

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}\}$$
$$\leq \min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}}^\theta)\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}}^\vartheta)\}\} + \theta + \vartheta$$
$$= \rho(\hat{\mathcal{A}}^\theta, \hat{\mathcal{B}}^\vartheta) + \theta + \vartheta.$$

Theorem 3.7

If $\rho: 2^{\tilde{\mathbb{X}}} \times 2^{\tilde{\mathbb{X}}} \longrightarrow [0, \infty]$ is an Dh-functional on $\tilde{\mathbb{X}}$ and for all $\varepsilon, \gamma \in \Re^+$. Then the family t_ε defined by $t_\varepsilon(\hat{\mathcal{A}}) = \min\{\hat{\mathcal{A}}^\varepsilon, \hat{\mathcal{A}}^\gamma\}$ is a tower on $\tilde{\mathbb{X}}$.

Proof

(To1) Let $\hat{\mathcal{A}} \in 2^{\tilde{\mathbb{X}}}$, let $\varepsilon \in \Re^+$. Suppose that $\tilde{a} \in \hat{\mathcal{A}}$, since $\hat{\mathcal{A}} \subseteq \hat{\mathcal{A}}^\varepsilon$ such that $t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{A}}^\varepsilon$.
We have $\hat{\mathcal{A}}^\varepsilon \subseteq t_\varepsilon(\hat{\mathcal{A}})$.

(To2) Let $\varepsilon \in \Re^+ \hat{\mathcal{A}} \in 2^{\tilde{\mathbb{X}}}$. If $\hat{\mathcal{A}} = \emptyset$, we obtain

$$t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{A}}^\varepsilon = \{\tilde{a} \in \tilde{\mathbb{X}} : \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon\}.$$

But $\hat{\mathcal{A}} = \emptyset$, then $\rho(\{\tilde{a}\}, \emptyset) = \infty$. Hence $t_\varepsilon(\emptyset) = \emptyset$.

(To3) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}$, let $\varepsilon \in \Re^+$.
Assume $\tilde{a} \in t_\varepsilon(\hat{\mathcal{A}} \cup \hat{\mathcal{B}})$, then $\tilde{a} \in (\hat{\mathcal{A}} \cup \hat{\mathcal{B}})^\varepsilon$. So that

$$(\hat{\mathcal{A}} \cup \hat{\mathcal{B}})^\varepsilon = \{\tilde{a} \in \tilde{\mathbb{X}} :\ \rho(\{\mathfrak{x}\}, \hat{\mathcal{A}} \cup \hat{\mathcal{B}}) \leq \varepsilon\}$$
$$= \{\tilde{a} \in \tilde{\mathbb{X}} : \min\{\rho(\{\tilde{a}\}, \hat{\mathcal{A}}), \rho(\{\tilde{a}\}, \hat{\mathcal{B}})\} \leq \varepsilon\}$$
$$= \{\tilde{a} \in \tilde{\mathbb{X}} : \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon, \rho(\{\tilde{a}\}, \hat{\mathcal{B}}) \leq \varepsilon\}$$
$$= \{\tilde{a} \in \tilde{\mathbb{X}} :\ \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon\} \cup \{\tilde{a} \in \tilde{\mathbb{X}} :\ \rho(\{\tilde{a}\}, \hat{\mathcal{B}}) \leq \varepsilon\}$$
$$= t_\varepsilon(\hat{\mathcal{A}}) \cup t_\varepsilon(\hat{\mathcal{B}})$$

(To4) Let $\hat{\mathcal{A}} \in 2^{\tilde{\mathbb{X}}}$, let $\varepsilon, \gamma \in \Re^+$. Assume $\tilde{a} \in t_\varepsilon(t_\gamma(\hat{\mathcal{A}}))$, then $\tilde{a} \in ((\mathcal{A})^{(\gamma)})^{(\varepsilon)}$ such that

$$((\hat{\mathcal{A}})^{(\gamma)})^{(\varepsilon)} = \{\tilde{a} \in \tilde{\mathbb{X}} : \rho(\{\tilde{a}\}, \hat{\mathcal{A}}^\gamma) \leq \varepsilon\}$$
$$= \{\tilde{a} \in \tilde{\mathbb{X}} : \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon + \gamma\}$$

$= t_{\varepsilon+\gamma}(\hat{\mathcal{A}})$

(To5) Let $\hat{\mathcal{A}} \in 2^{\widetilde{\mathbb{X}}}$, $\varepsilon \in \Re^+$. Assume that $\tilde{a} \in t_\varepsilon(\hat{\mathcal{A}})$, then $\tilde{a} \in \hat{\mathcal{A}}^\varepsilon$ such that $\hat{\mathcal{A}}^\varepsilon = \{\tilde{a} \in \widetilde{\mathbb{X}} : \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \le \varepsilon\}$. Take $\varepsilon < \gamma$, since $t_\varepsilon(\hat{\mathcal{A}}) = \min\{\hat{\mathcal{A}}^\varepsilon, \hat{\mathcal{A}}^\gamma\}$. Then $\tilde{a} \in \bigcap_{\varepsilon<\gamma} t_\gamma(\hat{\mathcal{A}})$.

Lemma 3.8

Let $\widetilde{\mathbb{X}}$ be a set and $\rho: 2^{\widetilde{\mathbb{X}}} \times 2^{\widetilde{\mathbb{X}}} \to [0,\infty]$, then following statements are equivalent:

(1) ρ is an Dh-functional on $\widetilde{\mathbb{X}}$,

(2) ρ satisfies (h1) - (h4) and

(h5.1) $\forall \hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}, \forall \varepsilon \in [0,\infty]: \ \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}^{(\varepsilon)}) + \varepsilon$,

(3) ρ satisfies (h1) - (h4) and (h 5.2) $\forall \hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{\widetilde{\mathbb{X}}}: \ \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \rho(\hat{\mathcal{A}}, \hat{\mathcal{C}}) + \sup_{c\in\hat{\mathcal{C}}} \rho(\{c\}, \hat{\mathcal{B}})$,

(4) ρ satisfies (h1) - (h4) and (h5.3) $\forall \mathcal{A}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}} \in 2^{\widetilde{\mathbb{X}}}$:

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \rho(\hat{\mathcal{D}}, \hat{\mathcal{C}}) + \sup_{c\in\hat{\mathcal{C}}} \rho(\{c\}, \hat{\mathcal{B}}) + \sup_{d\in\hat{\mathcal{D}}} \rho(\{d\}, \hat{\mathcal{A}}).$$

Proof

(1)⟹(2) From (1), we can deduce that (h1)-(h4) is satisfied. We demonstrate that (h5.1) is valid. Since $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \rho(\hat{\mathcal{A}}^\varepsilon, \hat{\mathcal{B}}^\eta) + \varepsilon + \eta \ \ \forall \eta \in [0,\infty]$, but $\hat{\mathcal{A}} \subseteq \hat{\mathcal{A}}^\varepsilon, \hat{\mathcal{B}} \subseteq \hat{\mathcal{B}}^\eta$.

Thus $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) + \varepsilon + \eta$, since $\eta \in [0,\infty]$ *and* $\hat{\mathcal{B}} \coloneqq \hat{\mathcal{B}}^\varepsilon$. We obtain

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}^\varepsilon) + \varepsilon.$$

(2)⟹(1) Suppose that ρ is satisfy (h1) - (h4), we show that (h5).

Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}, \varepsilon, \gamma \in [0,\infty]$. Take $\hat{\mathcal{A}} \coloneqq \hat{\mathcal{A}}^\varepsilon$ and $\hat{\mathcal{B}} \coloneqq \hat{\mathcal{B}}^\gamma$, we have

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \rho(\mathcal{A}^\varepsilon, \hat{\mathcal{B}}^\eta) + \varepsilon + \eta$$

(3) ⟹ (4) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}} \in 2^{\widetilde{\mathbb{X}}}$, put $\hat{\mathcal{A}} \coloneqq \hat{\mathcal{D}}$, we have (4).

(4) ⟹ (3) Given $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{\widetilde{\mathbb{X}}}$. Put $\hat{\mathcal{D}} \subset \hat{\hat{\mathcal{A}}}$, we obtain (h5.2).

(1) ⟹ (4) Suppose that (1) is satisfy and let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}} \in 2^{\widetilde{\mathbb{X}}}$. Take

$$\varepsilon = \inf\{\tau \in [0,\infty]: \hat{\mathcal{D}} \subset \hat{\mathcal{A}}^{(\tau)}\},$$

and

$$\gamma = \inf\{\tau \in [0,\infty]: \hat{\mathcal{C}} \subset \hat{\mathcal{A}}^{(\tau)}\}$$

Then

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \rho(\hat{\mathcal{A}}^\varepsilon, \hat{\mathcal{B}}^\gamma) + \varepsilon + \gamma \le \rho(\hat{\mathcal{D}}, \hat{\mathcal{C}}) + \sup_{c\in\hat{\mathcal{C}}} \rho(\{c\}, \hat{\mathcal{B}}) + \sup_{d\in\hat{\mathcal{D}}} \rho(\{d\}, \hat{\mathcal{A}}).$$

(4) ⟹ (1) Assume that $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}$, $\varepsilon, \gamma \in [0,\infty]$, Take $\hat{\mathcal{D}} \coloneqq \hat{\mathcal{A}}^{(\varepsilon)}, \hat{\mathcal{C}} \coloneqq \hat{\mathcal{B}}^{(\gamma)}$

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \rho(\hat{\mathcal{D}}, \hat{\mathcal{C}}) + \sup_{c\in\hat{\mathcal{C}}} \rho(\{c\}, \hat{\mathcal{B}}) + \sup_{d\in\hat{\mathcal{D}}} \rho(\{d\}, \hat{\mathcal{A}})$$
$$\le \rho(\hat{\mathcal{A}}^\varepsilon, \hat{\mathcal{B}}^\gamma) + \varepsilon + \gamma.$$

Definition 3.9

Assume that $\widetilde{\mathbb{X}}$ is a set and that for all $\varepsilon \in \Re^+$, If the following requirements are satisfied that then the function $t_\varepsilon: 2^{\widetilde{\mathbb{X}}} \to 2^{\widetilde{\mathbb{X}}}$ is called an Lo-*tower on* $\widetilde{\mathbb{X}}$ are:

(Lo-T1) $\forall \varepsilon \in \Re^+ : \ t_\varepsilon(\hat{\mathcal{A}}) = \max_{\gamma>\varepsilon} t_\gamma(\hat{\mathcal{A}})$,

(Lo-T2) $\forall \hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{\widetilde{\mathbb{X}}}, \forall \varepsilon, \gamma \in \Re^+ : \ t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}} \wedge t_\gamma(\{\tilde{b}\}) = \hat{\mathcal{C}} \ \ \forall \tilde{b} \in \hat{\mathcal{B}}$, then $t_{\varepsilon+\gamma}(\hat{\mathcal{A}}) = \hat{\mathcal{C}}$.

Theorem 3.10

Given a set $\widetilde{\mathbb{X}}$, then

(1) If ρ is an Dh-functional on $\widetilde{\mathbb{X}}$, and for all $\forall \varepsilon \in \Re^+$ such that $t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}}$ if and only if $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \varepsilon$, *then* $(t_\varepsilon)_{\varepsilon\in R^+}$ *is an* Lo-*tower on* $\widetilde{\mathbb{X}}$,

(2) If $(t_\varepsilon)_{\varepsilon\in R^+}$ is an Lo-tower on $\mathbb{X}$, then $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \coloneqq \inf\{\varepsilon \in \Re^+: \ t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}}\}$ is a Dh-functional on $\widetilde{\mathbb{X}}$.

Proof

(1) Let ρ be Dh-functional and let $\varepsilon \in R^+$ such that $t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}} \Leftrightarrow \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \le \varepsilon$.

(Lo-T1) Let $\tilde{a} \in t_\varepsilon(\hat{\mathcal{A}})$, $\tilde{a} \in \hat{\mathcal{B}}$ then $\tilde{a} \in \hat{\mathcal{A}}^\varepsilon$ such that $\hat{\mathcal{A}}^\varepsilon = \{\tilde{a} \in \widetilde{\mathbb{X}} : \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \le \varepsilon\}$.

Since $\rho(\hat{\mathcal{A}},\hat{\mathcal{B}}) \le \varepsilon$, then for $\varepsilon < \gamma$, $\max\{\tilde{a} \in \tilde{\mathbb{X}} : \rho(\{\tilde{a}\},\hat{\mathcal{A}}) \le \varepsilon < \gamma\}$. Thus $\tilde{a} \in \max\{\tilde{a} \in \tilde{\mathbb{X}} : \rho(\{\tilde{a}\},\hat{\mathcal{A}}) \le \gamma\}$, then $\tilde{a} \in \max_{\gamma>\varepsilon} t_\gamma(\hat{\mathcal{A}})$. So $t_\varepsilon(\hat{\mathcal{A}}) = \max_{\gamma>\varepsilon} t_\gamma(\hat{\mathcal{A}})$.

(Lo-T2) Let $\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{\mathcal{C}} \in 2^{\tilde{\mathbb{X}}}$, $let\ \varepsilon,\gamma \in R^+\ and\ \tilde{b} \in \hat{\mathcal{B}}$ such that $t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}} \wedge t_\gamma(\{\tilde{b}\}) = \hat{\mathcal{C}}$ if and only if

$$\rho(\hat{\mathcal{A}},\hat{\mathcal{B}}) \le \varepsilon \wedge \rho(\{\tilde{b}\},\hat{\mathcal{C}}) \le \gamma.$$

Suppose that $\tilde{a} \in t_{\varepsilon+\gamma}(\hat{\mathcal{A}})$ implies $\tilde{a} \in \mathcal{A}^{\varepsilon+\gamma} = \{\tilde{a} \in \tilde{\mathbb{X}} : \rho(\{\tilde{a}\},\hat{\mathcal{A}}) \le \varepsilon+\gamma\}$, but ρ is symmetric. Then $\rho(\hat{\mathcal{B}},\hat{\mathcal{A}}) \le \varepsilon$ from condition (h4). We obtain $\rho(\hat{\mathcal{B}},\hat{\mathcal{A}} \cup \hat{\mathcal{C}}) \le \min\{\varepsilon,\gamma\}$, take $\{\tilde{a}\} \subseteq \hat{\mathcal{A}} \cup \hat{\mathcal{B}}, \gamma < \varepsilon$.

So that $\rho(\{\hat{b}\},\{\tilde{a}\}) \le \gamma$ $\hat{b} \in \hat{\mathcal{B}}$, hence $\rho(\{\tilde{a}\},\{\hat{b}\}) \le \gamma$. Therefore $\tilde{a} \in t_\gamma(\{\tilde{b}\}) = \hat{\mathcal{C}}$, implies that $\tilde{a} \in \hat{\mathcal{C}}$.

(2) (h1) Let $\hat{\mathcal{A}},\hat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}$, put $\hat{\mathcal{A}} = \hat{\mathcal{A}}^\varepsilon$, $\hat{\mathcal{B}} = \hat{\mathcal{B}}^\varepsilon$

$$\rho(\hat{\mathcal{A}},\hat{\mathcal{B}}) = \inf\{\varepsilon \in \mathfrak{R}^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}}\} = \inf\{\varepsilon \in \mathfrak{R}^+ : \hat{\mathcal{A}}^\varepsilon = \hat{\mathcal{B}}\}$$
$$= \inf\{\varepsilon \in R^+ : \hat{\mathcal{A}} = \hat{\mathcal{B}}^\varepsilon\} = \inf\{\varepsilon \in R^+ : t_\varepsilon(\hat{\mathcal{B}}) = \hat{\mathcal{A}}\} = \rho(\hat{\mathcal{B}},\hat{\mathcal{A}})$$

(h2) Let $\hat{\mathcal{A}},\hat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}$, if $\hat{\mathcal{A}} = \emptyset$, we have

$$\rho(\hat{\mathcal{A}},\hat{\mathcal{B}}) = \inf\{\varepsilon \in \mathfrak{R}^+ : t_\varepsilon(\emptyset) = \hat{\mathcal{B}}\} = \inf\{\varepsilon \in \mathfrak{R}^+ : \emptyset = \hat{\mathcal{B}}\} = \infty$$

If $B = \emptyset$, we have

$$\rho(\hat{\mathcal{A}},\hat{\mathcal{B}}) = \inf\{\varepsilon \in \mathfrak{R}^+ : t_\varepsilon(\hat{\mathcal{A}}) = \emptyset\} = \infty$$

(h3) Let $\rho(\hat{\mathcal{A}},\hat{\mathcal{B}}) = 0$, $\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}}$ so $\{0\} = \inf\{\varepsilon \in R^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}}\}$. Then $\inf_{\tilde{a}\in\hat{\mathcal{A}}} \rho(\{\tilde{a}\},\hat{\mathcal{B}}) = 0$ and $\inf_{\hat{b}\in\hat{\mathcal{B}}} \rho(\hat{\mathcal{A}},\{\tilde{b}\}) = 0$, so that $\tilde{a} \in \hat{\mathcal{B}}$ and $\tilde{a} \in \hat{\mathcal{A}}$. Hence $\hat{\mathcal{A}} \cap \hat{\mathcal{B}} \ne \emptyset$

(h4) Let $\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{\mathcal{C}} \in 2^{\tilde{\mathbb{X}}}$,

$$\rho(\hat{\mathcal{A}},\hat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \inf\{\varepsilon \in \mathfrak{R}^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}} \cup \hat{\mathcal{C}}\}$$
$$= \inf\{\varepsilon \in \mathfrak{R}^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}} \cup t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{C}}\}$$
$$= \inf\{\varepsilon \in \mathfrak{R}^+ : \min\{t_\varepsilon(\mathcal{A}) = \hat{\mathcal{B}}, t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{C}}\}\}$$
$$= \min[\inf\{\varepsilon \in \mathfrak{R}^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}}\}, \inf\{\varepsilon \in R^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{C}}\}]$$
$$= \min\{\rho(\hat{\mathcal{A}},\hat{\mathcal{B}}), \rho(\hat{\mathcal{A}},\hat{\mathcal{C}})\}$$

(h5) Let $\hat{\mathcal{A}},\hat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}$, let $\theta,\varepsilon \in [0,\infty]$

$$\rho(\hat{\mathcal{A}},\hat{\mathcal{B}}) = \inf\{\varepsilon \in \mathfrak{R}^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}}\}$$
$$= \inf\{\varepsilon \in \mathfrak{R}^+ : \hat{\mathcal{B}} = \rho(\{\tilde{a}\},\hat{\mathcal{A}}) \le \varepsilon\}$$
$$\le \rho(\hat{\mathcal{A}},\hat{\mathcal{B}}^\varepsilon) + \varepsilon + \theta.$$

4 CONCLUSION

In order to study the concept of a new kind of proximit space, the relationship between Dh-functional and the tower, as well as certain fundamental facts and instances, are used. It should also include proximit space-related theorems. We also show the relation between proximit and metric space.

REFERENCES

[1] R. K. Abbas and B Y Hussein, “New Results of Completion Normed Approach Space,” AI conf. Proc. of the 2nd International Scientific conferences of Pure Sciences University of Al-Qadisiyah, 2021.

[2] R. K. Abbas and B Y Hussein, “A new kind of topological vector space: Topological approach vector space,” AIP conf. Proc. Of the 3rd International Scientific conferences of Al-Kafeel university, 2022.

[3] H. Herrlich, “Topologie II: Uniforme Räume, ” Verlag, Berlin, 1988.

[4] E. Lowen and R. Lowen, “A quasitopos containing CONV and MET as full subcategories,” Internat. J . Math. and Math. Sci., VOI. 11, 1988 , pp. 417-433.

[5] R. Lowen, “Approach spaces: a common supercategory of TOP and MET,” Math. Nachr., VOI. 14, 1989, pp.183-226.

[6] R. Lowen and K. Robey, “Completion of product of metric spaces,” Quart. J. Math. Oxford, VOI. 43, 1991 , pp. 319-338.

[7] R. Lowen, M. Sioen, “The Wijsman and Attouch-Wets topologies on hyperspaces revisited,” Topology Appl., VOI. 70, 1996, pp.179-197.
[8] R. Lowen, “Approach spaces: the missing link in the topologies- uniformities-metrics,” Oxford University Press, 1997.
[9] R. Lowen, M. Sioen, “Proximal hypertopologies revisited, ” Set-Valued Anal.,” 1998, pp.1-19.
[10] R. Lowen, M. Sioen, “Weak representations of quantified hyperspace structures, ” Topology Appl. VOI. 104 , 2000, pp. 169-179.
[11] R. Lowen, M. Sioen, “Approximations in functional analysis,” Result. Math., VoI. 37, 2000, pp. 345-372.
[12] R. Lowen, “Index Analysis: Approach Theory at Work,” Springer Monographs in Mathematics, Springer Verlag, 2015.
[13] S.A. Naimpally, B.D. Warrack, “Proximity Spaces, ” Cambridge University Press, Cambridge, 1970.
[14] M. Sioen, W. Van Den Haute, “An Alternative Description of Approach Spaces Via Approach Cores, ” Appl. Categor. Struct., 2016, pp. 545-558.
[15] Wei Li, Dexue Zhang, “Sober metric approach spaces, ” Topology and its Applications, VOI. 233, 2018, pp. 67-88.