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A New Kind of Topological Vector Space Via Proximit Structure

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ABSTRACT

The concepts of Dh-contraction, proximit semigroup, proximit group, proximit vector space and topological proximit space are introduced. There are a few instances and properties provided. Convergence and sequential Dh-contraction are both specified, in addition to the terms Dh-contraction and sequential Dh-contraction are interchangeable.

KEYWORDS: Dh-contraction, proximit group, proximit vector space, proximit space

1 INTRODUCTION

Approach spaces have some applicative backgrounds in all major of mathematics containing probability theory [10], domain theory [11], group theory [8] and vector spaces [9]. It is commonly known that the category Met of metric spaces and inexpensive maps lacks infinite products and coproducts. In order to address this issue, Robert Lowen [5] created approach spaces in 1989. These spaces are a generalization of metric and topology based on a distance that can be described in a variety of comparable ways, such as in terms of limit, gauge, and distance function between points and sets. Approach spaces [7,12] can be used to define closure operators in topology, extended pseudo-quasi-metrics for defining coarser topologies, and filter limit points, respectively. R. Lowen and K. Robeys [6] established some basic definitions and theorems in approach space which is a topological and metric space extension. More information about approach space can be found in [4]. M. Sion and W. Haute [13] investigated an alternative notion of approach spaces based on approach cores. [14] presented the relationship between approach space and metric space, which is similar to the relationship between topological and ordered space and he presented a sober metric approach space. A new kind of topololgical vector space which is called approach vector space made by [2]. Furthermore, Abbas and Hussein worked the completion normed approach space in [1]. This research looks into the regularity of functionals between sets. In section 2, we introduce some concepts of proximit space and we look at some of the findings and discuss the concept of Dh-contraction. Section 3 introduces the concept of proximit semigroup and proximit group. We also provide basic features and examples. Finally, topological proximit vector space, Dh-contraction, and proximit vector space are demonstrated.

2 PRELIMINARIES

Definition 2.1:[7]

A collection of functions $t_\varepsilon: 2^{\mathbb{X}} \to 2^{\mathbb{X}}$, $\varepsilon \in R^+$ is called a tower on $\mathbb{X}$ if the following conditions are satisfied :

(1) $\forall \hat{\mathcal{A}} \in 2^{\mathbb{X}}, \forall \varepsilon \in R^+: \hat{\mathcal{A}} \subset t_\varepsilon(\hat{\mathcal{A}})$,

(2) $\forall \varepsilon \in R^+: t_\varepsilon(\emptyset) = \emptyset$

(3) $\forall \hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\mathbb{X}}, \forall \varepsilon \in R^+: t_\varepsilon(\hat{\mathcal{A}} \cup \hat{\mathcal{B}}) = t_\varepsilon(\hat{\mathcal{A}}) \cup t_\varepsilon(\hat{\mathcal{B}})$,

(4) $\forall \hat{\mathcal{A}} \in 2^{\mathbb{X}}, \forall \varepsilon, \gamma \in R^+: t_\varepsilon(t_\gamma(\hat{\mathcal{A}})) \subset t_{\varepsilon+\gamma}(\hat{\mathcal{A}})$,

(5) $\forall \hat{\mathcal{A}} \in 2^{\mathbb{X}}, \forall \varepsilon \in R^+: t_\varepsilon(\hat{\mathcal{A}}) = \bigcap_{\varepsilon<\gamma} t_\gamma(\hat{\mathcal{A}})$,

From (3) and (5), Then $\forall \hat{\mathcal{A}} \subset \hat{\mathcal{B}} \subset \mathbb{X}, \forall \alpha \le \beta \in R^+: t_\alpha(\hat{\mathcal{A}}) \subset t_\beta(\hat{\mathcal{B}})$.

Definition 2.2:[7]

An operator $cl: 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ that determines the qualities of a pretopology on a set $\mathbb{X}$ for all $\mathcal{A}, \mathcal{B} \in 2^{\mathbb{X}}$ is determined by an operator that determines the properties of an operator on a set.

1. $\hat{\mathcal{A}} \subset cl(\hat{\mathcal{A}})$
2. $cl(\emptyset) = \emptyset$,
3. $cl(\hat{\mathcal{A}} \cup \hat{\mathcal{B}}) = cl(\hat{\mathcal{A}}) \cup cl(\hat{\mathcal{B}})$.

This operator is said to be a pretopological closure operator. We call a set $\mathbb{X}$ with a pretopology a *pretopological space.*

Definition 2.3: [3]

A functional $\rho: 2^{\mathbb{X}} \times 2^{\mathbb{X}} \to [0, \infty]$ is said to be an Dh-functional on $\mathbb{X}$ ($\mathbb{X}$ is a set) if and only if the following conditions are satisfies:

(G1) $\forall\ \hat{\mathcal{A}}, \mathcal{B} \in 2^{\mathbb{X}}:\ \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \rho(\hat{\mathcal{B}}, \hat{\mathcal{A}})$,

(G2) $\forall\ \hat{\mathcal{A}}, \mathcal{B} \in 2^{\mathbb{X}}\ ,\ \hat{\mathcal{A}} = \emptyset\ or\ \hat{\mathcal{B}} = \emptyset\ \Rightarrow\ \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \infty$,

(G3) $\forall\ \hat{\mathcal{A}}, \mathcal{B} \in 2^{\mathbb{X}}, \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}})\ =\ 0\ \ \Rightarrow \hat{\mathcal{A}} \cap \hat{\mathcal{B}} \neq\ \emptyset$,

(G4) $\forall \hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{\mathbb{X}}:\ \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \min\{\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}), \rho(\hat{\mathcal{A}}, \hat{\mathcal{C}})\}$,

(G5) $\forall \hat{\mathcal{A}}, \mathcal{B} \in 2^{\mathbb{X}}, \forall \varepsilon, \eta\ \in\ [0, \infty]:\ \rho(\hat{\mathcal{A}}, \mathcal{B}) \le\ \rho(\hat{\mathcal{A}}^\varepsilon, \mathcal{B}^\eta) + \varepsilon\ + \eta$.

For every $\hat{\mathcal{A}}\ \in 2^{\mathbb{X}}, \varepsilon \in [0, \infty]$, we write $t_\varepsilon(\hat{\mathcal{A}}) := \{\mathfrak{x} \in \mathbb{X} \mid \rho(\{\mathfrak{x}\}, \hat{\mathcal{A}}) \le \varepsilon\}$. Therefore the triple $(\mathbb{X}, \rho, t_\varepsilon)$ is called proximit space.

Example 2.4: [3]

Let $\mathbb{X} = [0, \infty]$, define $\rho: 2^{[0,\infty]} \times 2^{[0,\infty]} \to [0, \infty]$ by

$$\rho(\hat{\mathcal{A}}, \mathcal{B}) = \begin{cases} 0 & \hat{\mathcal{A}}, \hat{\mathcal{B}}\ unbounded \\ \infty & \hat{\mathcal{A}}, \hat{\mathcal{B}}\ bounded \\ \inf\limits_{n \in \hat{\mathcal{A}}} \inf\limits_{m \in \hat{\mathcal{B}}} |n - m| & n, m < \infty \end{cases}$$

Then $(\mathbb{X}, \rho, t_\varepsilon)$ is proximit space.

Definition 2.5

A function $f: (\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon) \to (\mathbb{Y}, \rho_{\mathbb{Y}}, t_\varepsilon)$ such that $(\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon)$ and $(\mathbb{Y}, \rho_{\mathbb{Y}}, t_\varepsilon)$ are proximit space is called *Dh-contraction* if $f\left(t_{\varepsilon_{\mathbb{X}}}(\hat{\mathcal{A}})\right) \subseteq t_{\varepsilon_{\mathbb{Y}}}(f(\mathcal{A}))\ \ \forall \hat{\mathcal{A}} \subseteq \mathbb{X}, \forall \varepsilon \in R^+$.

Proposition 2.6

Suppose that $(\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon)$ is proximit spaces. Then identity map $I_x: (\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon) \to (\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon)$ $\forall \hat{\mathcal{A}} \subseteq \mathbb{X}$, $\forall \varepsilon \in R^+$ is Dh-contraction.

Proof

Let $\hat{\mathcal{A}} \subseteq \mathbb{X},\ \varepsilon \in R^+$ such that $I_{\mathfrak{x}}(\hat{\mathcal{A}}) = \hat{\mathcal{A}}$. Let $\mathfrak{x} \in I_{\mathfrak{x}}\left(t_\varepsilon(\hat{\mathcal{A}})\right)$, then $\mathfrak{x} \in I_{\mathfrak{x}}\left(t_\varepsilon(\hat{\mathcal{A}})\right)$ implies that $\mathfrak{x} \in t_\varepsilon(\hat{\mathcal{A}})$. Hence $\mathfrak{x} \in \hat{\mathcal{A}}^\varepsilon$, it follows that $I_{\mathfrak{x}}(\mathfrak{x}) \in I_{\mathfrak{x}}(\hat{\mathcal{A}}^\varepsilon)$. But $\hat{\mathcal{A}}^\varepsilon := \hat{\mathcal{A}}$, so that $I_{\mathfrak{x}}(\mathfrak{x}) \in I_{\mathfrak{x}}(\hat{\mathcal{A}})$. From (1), we obtain $I_{\mathfrak{x}}(\mathfrak{x}) \in \left(I_{\mathfrak{x}}(\hat{\mathcal{A}})\right)^\varepsilon$. Then $\mathfrak{x} \in t_\varepsilon(I_{\mathfrak{x}}(\hat{\mathcal{A}}))$, So $I_{\mathfrak{x}}$ is Dh-contraction.

Proposition 2.7

Suppose that $(\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon), (\mathbb{Y}, \rho_{\mathbb{Y}}, t_\varepsilon)$ and $(\mathbb{Z}, \rho_{\mathbb{Z}}, t_\varepsilon)$ are proximit spaces. If the function $h: (\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon) \to (\mathbb{Y}, \rho_{\mathbb{Y}}, t_\varepsilon)$ and $j: (\mathbb{Y}, \rho_{\mathbb{Y}}, t_\varepsilon) \to (\mathbb{Z}, \rho_{\mathbb{Z}}, t_\varepsilon)$ are Dh-contraction, then $j \circ h$ is Dh-contraction.

Proof

Let $\hat{\mathcal{A}} \subseteq \mathbb{X},\ \varepsilon \in R^+$ such that $j \circ h: (\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon) \to (\mathbb{Z}, \rho_{\mathbb{Z}}, t_\varepsilon)$ be a function.

Let $\mathfrak{x} \in (j \circ h)\left(t_{\varepsilon_{\mathbb{X}}}(\hat{\mathcal{A}})\right)$, then $x \in (j(h\left(t_{\varepsilon_{\mathbb{X}}}(\hat{\mathcal{A}})\right))$. But h is Dh-contraction, it follows that $\mathfrak{x} \in j(t_{\varepsilon_{\mathbb{Y}}}(h(\hat{\mathcal{A}})))$. Also j is Dh-contraction, hence $\mathfrak{x} \in t_{\varepsilon_{\mathbb{Z}}}(j(h(\hat{\mathcal{A}})))$. Therefore $\mathfrak{x} \in t_{\varepsilon_{\mathbb{Z}}}(j \circ h)(\hat{\mathcal{A}})))$.

Then $(j \circ h)\left(t_{\varepsilon_{\mathbb{X}}}(\hat{\mathcal{A}})\right) \subseteq t_{\varepsilon_{\mathbb{Z}}}(j \circ h)(\hat{\mathcal{A}})))$, so that $j \circ h$ is Dh-contraction.

Proposition 2.8

If $(\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon), (\mathbb{Y}, \rho_{\mathbb{Y}}, t_\varepsilon)$ is a proximit space and $h: (\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon) \to (\mathbb{Y}, \rho_{\mathbb{Y}}, t_\varepsilon)$ is Dh-contraction, then the restriction $h|_S$ is Dh-contraction $\forall S \subseteq \mathbb{X}$.

Proof

Assume that $h: (\mathbb{X}, \rho_{\mathbb{X}}, t_\varepsilon) \to (\mathbb{Y}, \rho_{\mathbb{Y}}, t_\varepsilon)$ is Dh-contraction. We define restriction $j: S \longrightarrow (\mathbb{Y}, \rho_{\mathbb{Y}}, t_\varepsilon)$ as $j(v) = h(v)\ \ \forall v \in S$, assume $\mathfrak{x} \in j\left(t_{\varepsilon_{\mathbb{X}}}(\hat{\mathcal{A}})\right)$. Then $\mathfrak{x} \in h\left(t_{\varepsilon_{\mathbb{X}}}(\hat{\mathcal{A}})\right)$, but h is Dh-contraction. Hence $\mathfrak{x} \in t_{\varepsilon_{\mathbb{Y}}}(h(\hat{\mathcal{A}}))$, so $\mathfrak{x} \in t_{\varepsilon_{\mathbb{Y}}}(j(\hat{\mathcal{A}}))$. Thus $j\left(t_{\varepsilon_{\mathbb{X}}}(\hat{\mathcal{A}})\right) \subseteq t_{\varepsilon_{\mathbb{Y}}}(j(\hat{\mathcal{A}}))$, hence $h|_S$ is Dh-contraction.

3 Proximit Semi Group and Proximit Group

The terms "proximit semigroup," "proximit group," and "proximit subgroup" are defined in this section. Furthermore, instances and theorems are investigated.

Definition 3.1

A quadruple $(\mathbb{X}, \rho, t_\varepsilon, +)$ is said to be proximit semi group if and only if

1. $(\mathbb{X}, \rho, t_\varepsilon)$ is proximit space.
2. $(\mathbb{X}, +)$ is semi group.
3. $+: \mathbb{X} \otimes \mathbb{X} \to \mathbb{X}$ such that $(\mathfrak{x}, y) \longmapsto \mathfrak{x} + y$ is Dh-contraction.

Definition 3.2

A quadruple $(\mathbb{X}, \rho, t_\varepsilon, +)$ is said to be proximit group if and only if

1. $(\mathbb{X}, \rho, t_\varepsilon)$ is proximit space.
2. $(\mathbb{X}, +)$ is group.
3. $+: \mathbb{X} \otimes \mathbb{X} \to \mathbb{X}$ such that $(\mathfrak{x}, y) \longmapsto \mathfrak{x} + y$ is Dh-contraction.
4. $-: \mathbb{X} \longrightarrow \mathbb{X}$ such that $\mathfrak{x} \longmapsto -\mathfrak{x}$ is Dh-contraction.

Example 3.3

If R is the set of real numbers and $+$ is an addition binary operation with Dh-functional ρ, then $(R^n, \rho, t_\varepsilon, +)$ is proximit group.

Solve

1. $(R^n, \rho, t_\varepsilon)$ is proximit space.
2. $(R^n, +)$ is group.
3. We prove $+: R^n \otimes R^n \to R^n$ such that Dh-contraction is defined by $(\mathfrak{x}, \hbar) \longmapsto \mathfrak{x} + \hbar$ for all $\mathfrak{x} = (\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_n),\ \hbar = (\hbar_1, \hbar_2, \ldots, \hbar_n)$.

Let $\hat{\mathcal{A}}, \widehat{\mathcal{B}} \subseteq R^n, \varepsilon \in R^+$. Assume that $\mathfrak{x} \in f(t_\varepsilon(\hat{\mathcal{A}} + \widehat{\mathcal{B}}))$, then $f^{-1}(\mathfrak{x}) \in t_\varepsilon(\hat{\mathcal{A}} + \widehat{\mathcal{B}})$. So that $t_\varepsilon(\hat{\mathcal{A}} + \mathcal{B}) = \{f^{-1}(\mathfrak{x}) \in R^n : \rho(f^{-1}(\{\mathfrak{x}\}), \hat{\mathcal{A}} + \widehat{\mathcal{B}}) \leq \varepsilon\}$

$$= \left\{ f^{-1}(\mathfrak{x}) \in R^n : \inf_{f^{-1}(\mathfrak{x}) \in f^{-1}(\{\mathfrak{x}\})} \inf_{\hbar \in \hat{\mathcal{A}} + \widehat{\mathcal{B}}} d(f^{-1}(\mathfrak{x}), \hbar) \leq \varepsilon \right\}$$

$$= \left\{f^{-1}(\mathfrak{x}) \in R^n: \min\left\{\inf_{f^{-1}(\mathfrak{x})\in f^{-1}(\{\mathfrak{x}\})} \inf_{\hbar\in\hat{\mathcal{A}}} d(f^{-1}(\mathfrak{x}), a), \inf_{f^{-1}(\mathfrak{x})\in f^{-1}(\{\mathfrak{x}\})} \inf_{\hbar\in\widehat{\mathcal{B}}} d(f^{-1}(\mathfrak{x}), \hbar)\right\} \le \varepsilon \right\}$$
$$= \left\{\mathfrak{x} \in R^n: \min\left\{\inf_{\mathfrak{x}\in\{\mathfrak{x}\}} \inf_{f(\hbar)\in f(\hat{\mathcal{A}})} d(\mathfrak{x}, f(\hbar)), \inf_{\mathfrak{x}\in\{\mathfrak{x}\}} \inf_{f(\hbar)\in f(\widehat{\mathcal{B}})} d(\mathfrak{x}, f(\hbar))\right\} \le \varepsilon' \right\}$$
$$= \left\{\mathfrak{x} \in R^n: \inf_{\mathfrak{x}\in\{\mathfrak{x}\}} \inf_{f(\hbar)\in f(\hat{\mathcal{A}})} d(\mathfrak{x}, f(\hbar)) \le \varepsilon'\right\} + \left\{\mathfrak{x} \in R^n: \inf_{\mathfrak{x}\in\{\mathfrak{x}\}} \inf_{f(\hbar)\in f(\widehat{\mathcal{B}})} d(\mathfrak{x}, f(\hbar)) \le \varepsilon'\right\}.$$

Then $\rho(\{\mathfrak{x}\}, f(\hat{\mathcal{A}}) + f(\widehat{\mathcal{B}})) \le \varepsilon'$. Therefore $\mathfrak{x} \in t'_\varepsilon(f(\hat{\mathcal{A}}) + f(\widehat{\mathcal{B}}))$.

4. We show that $-: R^n \to R^n$ such that $\mathfrak{x} \mapsto -\mathfrak{x}$ is Dh-contraction. Let $\mathfrak{x} \in f(t_\varepsilon(\hat{\mathcal{A}}))$, then $f^{-1}(\mathfrak{x}) \in \hat{\mathcal{A}}^\varepsilon$.

So that $\hat{\mathcal{A}}^\varepsilon = \{f^{-1}(\mathfrak{x}) \in R^n : \rho(f^{-1}(\{\mathfrak{x}\}), \hat{\mathcal{A}}) \le \varepsilon\}$

$$= \left\{f^{-1}(\mathfrak{x}) \in R^n : \rho\left(-f^{-1}(\{\mathfrak{x}\}), -\hat{\mathcal{A}}\right) \le \varepsilon\right\}$$
$$= \{f^{-1}(\mathfrak{x}) \in R^n: \inf_{f^{-1}(\mathfrak{x})\in f^{-1}(\{x\})} \inf_{\hbar\in\hat{\mathcal{A}}} d(-f^{-1}(\mathfrak{x}), -\hbar) \le \varepsilon\}$$
$$= \{\mathfrak{x} \in R^n: \inf_{\mathfrak{x}\in\{\mathfrak{x}\}} \inf_{f(\hbar)\in f(\hat{\mathcal{A}})} d(-\{\mathfrak{x}\}, -f(\hbar)) \le \varepsilon'\}$$
$$= \left\{\mathfrak{x} \in R^n: \rho\left(\{\mathfrak{x}\}, f(\hat{\mathcal{A}})\right) \le \varepsilon'\right\}.$$

Thus $\mathfrak{x} \in t'_\varepsilon(f(\hat{\mathcal{A}}))$, so that $(R^n, \rho, t_\varepsilon, +)$ is proximit group.

Example 3.4

The set $C[a, b]$ of all continuous functions on the interval $[a, b]$. Then $(C[a, b], \rho, t_\varepsilon, +, .)$ is proximit group.

$$\rho(\hat{\mathcal{A}}, G) = \begin{cases} \infty & \hat{\mathcal{A}} = \emptyset \text{ or } G = \emptyset \\ \sup_{h\in\hat{\mathcal{A}}} \sup_{j\in G} |h(\mathfrak{x}) - j(\mathfrak{x})| & \hat{\mathcal{A}} \neq \emptyset \text{ and } G \neq \emptyset \end{cases}$$

Solve

Let $\hat{\mathcal{A}}, G, H \in 2^{C[a,b]}$

(G1) if $\hat{\mathcal{A}} \neq \emptyset, G \neq \emptyset$

$$\rho(\hat{\mathcal{A}}, G) = \sup_{h\in F} \sup_{j\in G} |h(\mathfrak{x}) - j(\mathfrak{x})| = \sup_{j\in G} \sup_{h\in\hat{\mathcal{A}}} |j(\mathfrak{x}) - h(\mathfrak{x})| = \rho(\hat{\mathcal{A}}, G)$$

(G2) If $\hat{\mathcal{A}} = \emptyset$,

$$\rho(\hat{\mathcal{A}}, G) = \sup_{h\in\emptyset} \sup_{j\in G} |\emptyset - j(\mathfrak{x})| = \infty$$

If $G = \emptyset$

$$\rho(\hat{\mathcal{A}}, G) = \sup_{h\in\hat{\mathcal{A}}} \sup_{j\in\emptyset} |h(\mathfrak{x}) - \emptyset| = \infty$$

(G3) If $\rho(\hat{\mathcal{A}}, G) = 0$, then $0 = \sup_{h\in\hat{\mathcal{A}}} \sup_{j\in G} |h(\mathfrak{x}) - j(\mathfrak{x})|$. It follows that $h(\mathfrak{x}) = j(\mathfrak{x})$, so $\hat{\mathcal{A}} \cap G \neq \emptyset$.

(G4) Let $\rho(\hat{\mathcal{A}}, G \cup H) = \sup_{h\in\hat{\mathcal{A}}} \sup_{j\in G\cup H} |h(\mathfrak{x}) - j(\mathfrak{x})|$

$$= \sup_{h\in\hat{\mathcal{A}}} \left(min\left\{ \sup_{j\in G} |h(\mathfrak{x}) - j(\mathfrak{x})|, \sup_{h\in\hat{\mathcal{A}}} \sup_{j\in H} |h(\mathfrak{x}) - j(\mathfrak{x})| \right\}\right)$$
$$= \min\left(\left\{\sup_{h\in\hat{\mathcal{A}}} \sup_{j\in G} |h(\mathfrak{x}) - j(\mathfrak{x})|, \sup_{h\in\hat{\mathcal{A}}} \sup_{j\in H} |h(\mathfrak{x}) - j(\mathfrak{x})|\right\}\right)$$
$$= \min\{\rho(\hat{\mathcal{A}}, G), \rho(\hat{\mathcal{A}}, H)\}$$

(G5) Let $\hat{\mathcal{A}}, G \in 2^{C[a,b]}, \varepsilon, \eta \in [0, \infty]$

$$\rho(\hat{\mathcal{A}}, G) = \sup_{h\in\hat{\mathcal{A}}} \sup_{j\in G} |h(\mathfrak{x}) - j(\mathfrak{x})| \le \sup_{h\in\hat{\mathcal{A}}} \sup_{j\in G} |h(\mathfrak{x}) - j(\mathfrak{x})| + \varepsilon + \eta = \rho(\hat{\mathcal{A}}^\varepsilon, G^\eta) + \varepsilon + \eta$$

2. $(C[a, b], +)$ is group.

3. We prove $+: C[a, b] \oplus C[a, b] \to C[a, b]$ such that $(h, j) \longmapsto h + j$ is Dh-contraction.

Let $\mathfrak{x} \in f(t_\varepsilon(\hat{\mathcal{A}} + G))$, then $f^{-1}(\mathfrak{x}) \in t_\varepsilon(\hat{\mathcal{A}} + G)$. It follows that

$t_\varepsilon(\hat{\mathcal{A}} + G) = \left\{f^{-1}(\mathfrak{x}) \in C[a, b]: \rho\left(f^{-1}(\{\mathfrak{x}\}), \hat{\mathcal{A}} + G\right) \le \varepsilon\right\}$

$$= \left\{ f^{-1}(\mathfrak{x}) \in \mathbb{X} : \sup_{f^{-1}(h(\mathfrak{x})) \in f^{-1}(\{\mathfrak{x}\})} \sup_{j \in \hat{\mathcal{A}}+G} |f^{-1}(h(\mathfrak{x})) - j(\mathfrak{x})| \leq \varepsilon \right\}$$

$$= \left\{ f^{-1}(\mathfrak{x}) \in \mathbb{X} : \min \left\{ \sup_{f^{-1}(h(\mathfrak{x})) \in f^{-1}(\{\mathfrak{x}\})} \sup_{j \in \hat{\mathcal{A}}} \{|f^{-1}(h(\mathfrak{x})) - j(\mathfrak{x})|\}, \sup_{f^{-1}(h(\mathfrak{x})) \in f^{-1}(\{\mathfrak{x}\})} \sup_{j \in G} \{|f^{-1}(h(\mathfrak{x})) - j(\mathfrak{x})|\} \right\} \leq \varepsilon \right\}$$

$$= \left\{ \mathfrak{x} \in \mathbb{X} : \min \left\{ \sup_{h(\mathfrak{x}) \in \{\mathfrak{x}\}} \sup_{f(j) \in f(\hat{\mathcal{A}})} \left|h(\mathfrak{x}) - f\big(j(\mathfrak{x})\big)\right|, \sup_{h(\mathfrak{x}) \in \{\mathfrak{x}\}} \sup_{f(j) \in f(G)} |h(\mathfrak{x}) - f(j(\mathfrak{x}))| \right\} \leq \varepsilon' \right\}$$

$$= \left\{ \mathfrak{x} \in \mathbb{X} : \left\{ \rho\big(\{\mathfrak{x}\}, f(\hat{\mathcal{A}}) + f(G)\big) \right\} \leq \varepsilon' \right\}$$

Then $\mathfrak{x} \in t'_{\varepsilon}(f(\hat{\mathcal{A}}) + f(G))$.

4. Let $\mathfrak{x} \in f(t_{\varepsilon}(\hat{\mathcal{A}}))$, then $f^{-1}(\mathfrak{x}) \in t_{\varepsilon}(\hat{\mathcal{A}})$. It follows that

$$t_{\varepsilon}(\hat{\mathcal{A}}) = \left\{ f^{-1}(\mathfrak{x}) \in \mathbb{X} : \rho\big(f^{-1}(\{\mathfrak{x}\}), \hat{\mathcal{A}}\big) \leq \varepsilon \right\}$$

$$= \left\{ f^{-1}(\mathfrak{x}) \in \mathbb{X} : \sup_{f^{-1}(h(\mathfrak{x})) \in f^{-1}(\{\mathfrak{x}\})} \sup_{j \in \hat{\mathcal{A}}} \left|-f^{-1}\big(h(\mathfrak{x})\big) - (-j(\mathfrak{x}))\right| \leq \varepsilon \right\}$$

$$= \left\{ f^{-1}(\mathfrak{x}) \in \mathbb{X} : \sup_{h(\mathfrak{x}) \in \{x\}} \sup_{f(j) \in f(\hat{\mathcal{A}})} |-h(\mathfrak{x}) + f(j(\mathfrak{x}))| \leq \varepsilon' \right\}$$

$$= \left\{ \mathfrak{x} \in \mathbb{X} : \left\{ \rho\left(\{\mathfrak{x}\}, f(\hat{\mathcal{A}})\right) \right\} \leq \varepsilon' \right\}$$

Then $\mathfrak{x} \in t'_{\varepsilon}(f(\hat{\mathcal{A}}))$, so that $(C[a,b], \rho, t_{\varepsilon}, +)$ is proximit group.

Definition 3.5

If $(\mathbb{X}, \rho, t_{\varepsilon}, +)$ is a proximit group and $\mathbb{Y} \subseteq \mathbb{X}$, then $\mathbb{Y}$ is said to be proximit sub group if the following conditions are satisfied:

1. $(\mathbb{Y}, \rho, t_{\varepsilon})$ is proximit space
2. $(\mathbb{Y}, +)$ is subgroup
3. $j: \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ such that $j(\mathfrak{x}, \mathscr{y}) = \mathfrak{x} * \mathscr{y}^{-1}$ is Dh-contraction.

Example 3.6

Let Z be the set of integers and if ρ is an Dh-functional defined by

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \begin{cases} \infty & \hat{\mathcal{A}} = \emptyset \text{ } or \text{ } \hat{\mathcal{B}} = \emptyset \\ \inf_{\mathfrak{x} \in \hat{\mathcal{A}}} \inf_{\mathscr{y} \in \hat{\mathcal{B}}} |\mathfrak{x} - \mathscr{y}| & \hat{\mathcal{A}} \neq \emptyset \text{ } and \text{ } \hat{\mathcal{B}} \neq \emptyset \end{cases}$$

Then $(Z, \rho, t_{\varepsilon}, +)$is a proximit subgroup of Q.

4 Proximit Vector Space

We present the idea of proximit vector space and show the new results.

Definition 4.1

A quintuple $(\mathbb{X}, \rho, t_{\varepsilon}, +, .)$ is called proximit vector space such that $\mathbb{X}$ is a non-empty set with two binary operations (addition and scalar multiplication) and triple $(\mathbb{X}, \rho, t_{\varepsilon})$ is Proximit space if satisfy the following conditions $\forall \mathscr{g}, \mathscr{g}^* \in \mathbb{X}$, ð, $\zeta \in$ field E.

1. $(\mathbb{X}, \rho, t_{\varepsilon}, +)$ is proximit group.
2. $\zeta . \mathscr{g} \in V$
3. $\zeta . (\mathscr{g} + \mathscr{g}^*) = \zeta\mathscr{g} + \zeta\mathscr{g}^*$
4. $(\mathscr{g} + \mathscr{g}^*)\zeta = \mathscr{g}.\zeta + \mathscr{g}^*.\zeta$
5. $(\vartheta.\zeta).\mathscr{g} = \vartheta(\zeta\mathscr{g})$
6. $1.\mathscr{g} = \mathscr{g}$

Example 4.2

Let $C[a,b]$ be set of all continuous functions on the interval $[a,b]$. Then $(C[a,b], \rho, t_{\varepsilon}, +, .)$ is proximit vector space.

Proof

1. $(C[a,b], \rho, t_\varepsilon, +)$ is proximit group by Example (3.4).
2. $\alpha . f(\mathfrak{x}) \in C[a,b]$.
3. Since $C[a,b]$ is linear space, then $\alpha\big(f(\mathfrak{x}) + g(\mathfrak{x})\big) = \alpha f(\mathfrak{x}) + \alpha g(\mathfrak{x})$
4. $\big(f(\mathfrak{x}) + g(\mathfrak{x})\big)\alpha = f(\mathfrak{x}).\alpha + g(\mathfrak{x}).\alpha$
5. $(\eth.\zeta).f(\mathfrak{x}) = \eth(\zeta f(\mathfrak{x}))$
6. $1.f(\mathfrak{x}) = f(\mathfrak{x})$.

Then $(C[a,b], \rho, t_\varepsilon, +, .)$ is proximit vector space.

Definition 4.3

A subset U of proximit vector space $(\mathbb{X}, \rho, t_\varepsilon, +, .)$ over a field E is said to be proximit subspace if the following conditions are holds.

1. U is subspace of vector space
2. $(U, \rho_U, t_\varepsilon)$ is proximit space.

Remark 4.4

We denoted topological space by $(\mathbb{X}, T)$ where T is a family of open sets such as associated closure operator, denoted by cl_T.

Definition 4.5

A topological space $(\mathbb{X}, T)$ that associated with a proximit space which its defined as a function $\rho_T: 2^{\mathbb{X}} \times 2^{\mathbb{X}} \to [0, \infty]$ such that

$$(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \mapsto \begin{cases} 0, & \hat{\mathcal{A}} \in cl_T(\hat{\mathcal{B}}) \\ \infty, & \hat{\mathcal{A}} \notin cl_T(\hat{\mathcal{B}}) \end{cases}$$

Proximit space of kind $(\mathbb{X}, T, \rho_T)$ for T of $\mathbb{X}$ is said to be a topological proximit space. Also, Dh-functional of type ρ_T is said to be a topological Dh-functional.

Proposition 4.6

Suppose that $(\Upsilon, \rho, t_\varepsilon, +, .)$ is proximit vector space on a field E. A topological proximit space induced by topology T_Υ , denoted by (Υ, T_Υ) satisfy the following:

1. The map $+: \Upsilon \times \Upsilon \to \Upsilon$ such that $(\mathfrak{x}, \mathcal{y}) \mapsto \mathfrak{x} + \mathcal{y}$ is Dh-contraction
2. The map $. : \mathrm{E} \times \Upsilon \longrightarrow \Upsilon$ such that $(\alpha, \mathfrak{x}) \mapsto \alpha\mathfrak{x}$ is Dh-contraction.

Proposition 4.7

Let (Υ, T) be topological space, then a function $\rho_T: 2^\Upsilon \times 2^\Upsilon \to [0, \infty]$ defined by

$$\rho_T(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \begin{cases} 0, & \hat{\mathcal{A}} \subseteq cl_T(\hat{\mathcal{B}}) \\ \infty, & \hat{\mathcal{A}} \nsubseteq cl_T(\hat{\mathcal{B}}) \end{cases}$$

is Dh-functional.

Proof

(1) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^\Upsilon$, if $\hat{\mathcal{A}} \subseteq cl_T(\hat{\mathcal{B}})$. Then $\rho_T(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = 0 = \rho_T(\hat{\mathcal{B}}, \hat{\mathcal{A}})$.
if $\hat{\mathcal{A}} \nsubseteq cl_T(\hat{\mathcal{B}})$. Then $\rho_T(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \infty = \rho_T(\hat{\mathcal{B}}, \hat{\mathcal{A}})$.

(2) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^\Upsilon$, if $\hat{\mathcal{A}} \subseteq cl_T(\hat{\mathcal{B}})$ and if $\hat{\mathcal{B}} = \emptyset$. Thus $cl_T(\emptyset) = \emptyset$, so $\rho_T(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \infty$.

(3) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^\Upsilon$, if $\rho_T(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = 0$. Then $\hat{\mathcal{A}} \subseteq cl_T(\hat{\mathcal{B}})$, but $\hat{\mathcal{A}} \subseteq cl_T(\hat{\mathcal{A}})$ and $\hat{\mathcal{B}} \subseteq cl_T(\hat{\mathcal{B}})$.
Thus $\hat{\mathcal{A}} \cap \hat{\mathcal{B}} \neq \emptyset$.

(4) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^\Upsilon$, Thus $cl_T(\hat{\mathcal{A}} \cup \hat{\mathcal{B}}) = c_l(\hat{\mathcal{A}}) \cup cl_T(\hat{\mathcal{B}})$, so

$$\rho_T(\hat{\mathcal{A}}, \hat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \min(\rho_T(\hat{\mathcal{A}}, \hat{\mathcal{B}}), \rho_T(\hat{\mathcal{A}}, \hat{\mathcal{C}})).$$

(5) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^\Upsilon, \varepsilon, \gamma \in [0, \infty]$, Thus $\hat{\mathcal{A}}^\varepsilon = cl_T(\hat{\mathcal{A}})$, $\hat{\mathcal{B}}^\gamma = cl_T(\hat{\mathcal{B}})$, $\hat{\mathcal{A}}^\varepsilon = \Upsilon$ and $\hat{\mathcal{B}}^\gamma = \Upsilon$. So that

$$\rho_T(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \leq \rho_T(\hat{\mathcal{A}}^\varepsilon, \hat{\mathcal{B}}^\gamma) + \varepsilon + \gamma$$

Theorem 4.8

If $(\Upsilon,\rho_\Upsilon,t_{\varepsilon\Upsilon})$ is proximit vector space, $\mathcal{M}$ is closed proximit subspace of $(\Upsilon,\rho_\Upsilon,t_{\varepsilon\Upsilon})$. Then $(\Upsilon/\mathcal{M}\,,\rho'_{\Upsilon/_{\mathcal{M}}},t_{\varepsilon\Upsilon/_{\mathcal{M}}})$ is proximit vector space such that $\rho'_\Upsilon/\mathcal{M}:2^{\Upsilon/\mathcal{M}}\times 2^{\Upsilon/\mathcal{M}}\to[0,\infty]$ is defined by $\rho'_{\Upsilon/_{\mathcal{M}}}(\mathcal{A},\widehat{\mathcal{B}})=\rho'_{\Upsilon/_{\mathcal{M}}}(\mathcal{P}+\mathcal{A},\mathcal{Q}+\widehat{\mathcal{B}})=\rho_\Upsilon(\hat{\mathcal{A}},\widehat{\mathcal{B}})$

Proof

Let $\hat{\mathcal{A}},\widehat{\mathcal{B}},\hat{\mathcal{C}}\subseteq\Upsilon$

1. $\rho'_{\Upsilon/_{\mathcal{M}}}(\hat{\mathcal{A}}+\mathcal{M},\widehat{\mathcal{B}}+\mathcal{M})=\rho_\Upsilon(\hat{\mathcal{A}},\widehat{\mathcal{B}})=\rho_\Upsilon(\widehat{\mathcal{B}},\hat{\mathcal{A}})=\rho'_{\Upsilon/_{\mathcal{M}}}(\widehat{\mathcal{B}}+\mathcal{M},\hat{\mathcal{A}}+\mathcal{M})$
2. If $\hat{\mathcal{A}}=\emptyset\ or\ \widehat{\mathcal{B}}=\emptyset$, then $\rho'_{\Upsilon/_{\mathcal{M}}}(\hat{\mathcal{A}}+\mathcal{M},\widehat{\mathcal{B}}+\mathcal{M})=\rho_\Upsilon(\hat{\mathcal{A}},\widehat{\mathcal{B}})=\infty$
3. If $\rho'_{\Upsilon/_{\mathcal{M}}}(\hat{\mathcal{A}}+\mathcal{M},\widehat{\mathcal{B}}+\mathcal{M})=0\,,then\ \rho_\Upsilon(\hat{\mathcal{A}},\widehat{\mathcal{B}})=0$. So that $\hat{\mathcal{A}}\cap\widehat{\mathcal{B}}\neq\emptyset$.
4. $\rho'_{\Upsilon/_{\mathcal{M}}}\left(\hat{\mathcal{A}}+\mathcal{M},(\widehat{\mathcal{B}}+\mathcal{M})\cup(\hat{\mathcal{C}}+\mathcal{M})\right)=\rho'_{\Upsilon/_{\mathcal{M}}}(\hat{\mathcal{A}}+\mathcal{M},(\widehat{\mathcal{B}}\cup\hat{\mathcal{C}})+\mathcal{M})\,,then$

$$\rho_\Upsilon(\hat{\mathcal{A}},\widehat{\mathcal{B}}\cup\mathcal{C})=\min\{\rho_\Upsilon(\hat{\mathcal{A}},\mathcal{B}),\rho_\Upsilon(\hat{\mathcal{A}},\mathcal{C})\}$$
$$=\min\{\rho'_{\Upsilon/_{\mathcal{M}}}(\hat{\mathcal{A}}+\mathcal{M},\widehat{\mathcal{B}}+\mathcal{M}),\rho'_{\Upsilon/_{\mathcal{M}}}(\hat{\mathcal{A}}+\mathcal{M},\hat{\mathcal{C}}+\mathcal{M})\}$$

5. Let $\hat{\mathcal{A}},\widehat{\mathcal{B}}\in 2^{\Upsilon/_{\mathcal{M}}}$, $\varepsilon,\gamma\in[0,\infty]$

$$\rho'_{\Upsilon/_{\mathcal{M}}}(\hat{\mathcal{A}}+\mathcal{M},\widehat{\mathcal{B}}+\mathcal{M})=\rho_\Upsilon(\hat{\mathcal{A}},\widehat{\mathcal{B}})\le\rho_\Upsilon\left(\hat{\mathcal{A}},\widehat{\widehat{\mathcal{B}}}\right)+\varepsilon+\gamma\le\rho'_{\Upsilon/_{\mathcal{M}}}(\hat{\mathcal{A}}^\varepsilon+\mathcal{M},\widehat{\mathcal{B}}^\gamma+\mathcal{M})+\varepsilon+\gamma.$$

Then $(\Upsilon/\mathcal{M}\,,\rho'_{\Upsilon/_{\mathcal{M}}},t_\varepsilon)$ is proximit space.

2. $(\Upsilon/\mathcal{M}\,,+)$ is group.
3. $+:\Upsilon/\mathcal{M}\times\Upsilon/\mathcal{M}\to\Upsilon/\mathcal{M}$ such that $(\mathfrak{x}+\mathcal{M},y+\mathcal{M})\longmapsto\mathfrak{x}+y+\mathcal{M}$

Let $\hat{\mathcal{A}},\widehat{\mathcal{B}}\subseteq{}^{\Upsilon}/_{\mathcal{M}}\,,\varepsilon\in R^+$. Suppose that $\mathfrak{x}\in f(t_\varepsilon(\hat{\mathcal{A}}+\widehat{\mathcal{B}}+\mathcal{M}))$, then $f^{-1}(\mathfrak{x})\in t_\varepsilon(\hat{\mathcal{A}}+\widehat{\mathcal{B}}+\mathcal{M})$ which it is equal to $\left\{f^{-1}(\mathfrak{x})\in\Upsilon/\mathcal{M}:\rho'_{\Upsilon/_{\mathcal{M}}}(\{f^{-1}(\mathfrak{x})\}+\mathcal{M},\hat{\mathcal{A}}+\widehat{\mathcal{B}}+\mathcal{M})\ \le\varepsilon\right\}$

$$=\{f^{-1}(\mathfrak{x})\in\Upsilon:\rho_\Upsilon(\{f^{-1}(\mathfrak{x})\},\hat{\mathcal{A}}+\widehat{\mathcal{B}})\le\varepsilon\}$$
$$=\{\mathfrak{x}\in\Upsilon:\rho_\Upsilon(\{x\},f(\hat{\mathcal{A}}+\widehat{\mathcal{B}}))\ \le\varepsilon\}$$
$$=\{\mathfrak{x}\in\Upsilon/\mathcal{M}:\rho'_{\Upsilon/_{\mathcal{M}}}(\{\mathfrak{x}\}+\mathcal{M},f(\hat{\mathcal{A}}+\widehat{\mathcal{B}})+\mathcal{M})\le\varepsilon\}.$$

Therefore $\mathfrak{x}\in t'_\varepsilon(f(\hat{\mathcal{A}}+\widehat{\mathcal{B}}))$,

4. Let $\hat{\mathcal{A}}\subseteq{}^{\Upsilon}/_{\mathcal{M}}\,,\varepsilon\in R^+$. Suppose that $\mathfrak{x}\in f(t_\varepsilon(\widehat{\mathcal{A}}+\mathcal{M}))$, then $f^{-1}(\mathfrak{x})\in t_\varepsilon(\widehat{\mathcal{A}}+\mathcal{M})$ which it is equal to

$$\{f^{-1}(\mathfrak{x})\in\Upsilon/\mathcal{M}:\rho'_{\Upsilon/_{\mathcal{M}}}(\{f^{-1}(\mathfrak{x})\}+\mathcal{M},\hat{\mathcal{A}}+\mathcal{M})\ \le\varepsilon\}=\{f^{-1}(\mathfrak{x})\in\Upsilon:\rho_\Upsilon(\{f^{-1}(\mathfrak{x})\},\hat{\mathcal{A}})\le\varepsilon\}$$
$$=\{\mathfrak{x}\in\Upsilon:\rho_\Upsilon(\{\mathfrak{x}\},f(\hat{\mathcal{A}}))\ \le\varepsilon\}$$
$$=\{\mathfrak{x}\in\Upsilon/\mathcal{M}:\rho'_{\Upsilon/_{\mathcal{M}}}(\{\mathfrak{x}\}+\mathcal{M},f(\hat{\mathcal{A}})+\mathcal{M})\le\varepsilon\}.$$

Therefore $\mathfrak{x}\in t'_\varepsilon(f(\hat{\mathcal{A}}))$, then $(\Upsilon/\mathcal{M}\,,\rho'_{\Upsilon/_{\mathcal{M}}},t_\varepsilon,+)$ is proximit group.

The remaining condition is direct.

Definition 4.9

If $(\Upsilon,\rho,t_\varepsilon)$ is proximit vector space and $\langle\mathfrak{X}_n\rangle$ is a sequence in $(\Upsilon,\rho,t_\varepsilon)$, then $\langle\mathfrak{X}_n\rangle$ is said to be convergent if $\exists\mathfrak{X}\in\Upsilon$ such that $\lim_{n\to\infty}\inf\rho(\{\mathfrak{X}_n\},\widehat{\mathcal{B}})=0$ and $\lim_{n\to\infty}\sup\rho(\{\mathfrak{X}_n\},\widehat{\mathcal{B}})=0$

Definition 4.10

If $(\Upsilon,\rho_\Upsilon,t_\varepsilon)$ and $(\mathcal{W},\rho_\mathcal{W},t_\varepsilon)$ are proximit spaces. We say that the function $\varphi:\Upsilon\longrightarrow\mathcal{W}$ is sequentially Dh-contraction if $\lim_{n\to\infty}\varphi\left(t_{\varepsilon_\Upsilon}(\{\mathfrak{X}_n\})\right)=0$ whenever $\lim_{n\to\infty}t_{\varepsilon_\mathcal{W}}(\varphi(\{\mathfrak{X}_n\}))=0$.

Theorem 4.11

If $(\Upsilon,\rho_\Upsilon,t_\varepsilon)$ and $(\mathcal{W},\rho_\mathcal{W},t_\varepsilon)$ are proximit spaces. Then a function $\varphi:\Upsilon\longrightarrow\mathcal{W}$ is Dh-contraction if and only if its sequentially Dh-contraction.

Proof

Let φ be an Dh-contraction, let $\{\mathfrak{X}_n\}$ be convergent sequence . Then $\lim_{n\to\infty} \inf \rho(\{\mathfrak{X}_n\}, \widehat{\mathcal{B}}) = 0$ and $\lim_{n\to\infty} \sup \rho(\{\mathfrak{X}_n\}, \widehat{\mathcal{B}}) = 0$, but $\varphi\left(t_{\varepsilon_{\Upsilon}}(\{\mathfrak{X}_n\})\right) \subseteq t_{\varepsilon_{\mathcal{W}}}(\varphi(\{\mathfrak{X}_n\}))$. Then $\lim_{n\to\infty} t_{\varepsilon_{\mathcal{W}}}(\varphi(\{\mathfrak{X}_n\})) = 0$ implies that $\lim_{n\to\infty} \varphi\left(t_{\varepsilon_{\Upsilon}}(\{\mathfrak{X}_n\})\right) = 0$. Thus φ is sequentially Dh-contraction.

Conversely, assume that φ is sequentially Dh-contraction. If φ is not Dh-contraction, then $\varphi\left(t_{\varepsilon_{\Upsilon}}(\{\mathfrak{X}_n\})\right) \nsubseteq t_{\varepsilon_{\mathcal{W}}}(\varphi(\{\mathfrak{X}_n\}))$ such that there exists $\mathscr{f} \in \varphi\left(t_{\varepsilon_{\Upsilon}}(\{\mathfrak{X}_n\})\right), \mathscr{f} \notin t_{\varepsilon_{\mathcal{W}}}(\varphi(\{\mathfrak{X}_n\}))$. Since φ is sequentially Dh-contraction, then $\lim_{n\to\infty} \varphi\left(t_{\varepsilon_{\Upsilon}}(\{\mathfrak{X}_n\})\right) = 0$ whenever $\lim_{n\to\infty} t_{\varepsilon_{\mathcal{W}}}(\varphi(\{\mathfrak{X}_n\})) = 0$ which is contradiction.

6 CONCLUSION

The regularity of functionals between sets is the subject of this study. The terms proximit semigroup and proximit group, as well as other proximit space notions, are introduced here. The notion of Dh-contraction is discussed as we look at some of the results. Furthermore, we include fundamental details and illustrations. Topological proximit vector space, Dh-contraction, and proximit vector space are studied.

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