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Toward new class of hyperbolic functions

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Abstract

In this paper, we introduce new class of hyperbolic functions, namely, symmetric hyperbolic Pell sine function and hyperbolic Pell cosine function. These functions, combine the concept of classical hyperbolic functions and recursive Pell sequence and Pell-Lucas sequence. We study some properties of these new hyperbolic functions and present some identities about these functions.

Keywords: Pell sequence, hyperbolic function, identity.

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1 Introduction

Hyperbolic functions are very widely used functions in mathematics, whose governing relations are similar to the governing relations of trigonometric functions, with the difference that trigonometric lines are defined with respect to a circle whose radius is one, but hyperbolic functions are defined with respect to isosceles hyperbolism. It can be seen that these functions appear in the scope of integrals, linear differential equations, and Laplace's equation. Among the applications of hyperbolic functions, one can mention the description of wave movement in elastic bodies, the shape of electric power transmission lines, temperature distribution in metal blades, tracking curves and the geometry of general relativity theory and other cases in various sciences(see [1], [10], [11] and [15]).

Number sequences like as Fibonacci sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence, ... have attracted many authors in the area of coding theory in computer sciences, signal processing, numerical analysis, combinatory theory and matrix theory. For instance, Bueno [2] studied circulant matrices involving Fibonacci numbers and drived the formula for its Euclidean norm and eigenvalues of that matrix. Bozkurt [4] computed the spectral norms of some matrices connected integer sequences such as Fibonacci, Lucas, Pell and Perrin numbers. For more information about Pell sequence, (k, h)-Pell sequence, Fibonacci

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sequence and some generalizations of these sequences we refer to [4], [6] - [9] and [12] - [14].

The authors in [12] considered Pell sequence and Pell-Lucas sequence and introduced some generalization of these sequence. Well-Known Pell sequence P_n is defined by

$$P_n = 2P_{n-1} + P_n; \quad P_0 = 0, P_1 = 1. \tag{1}$$

This recursive sequence has the characteristic equation $r^2 - 2r - 1 = 0$. This equation has two distinct roots $\theta = 1 + \sqrt{2}, \gamma = 1 - \sqrt{2}$. Hence, we can see that [12] the Binet's formula of Pell numbers is given by

$$P_n = \frac{\theta^n - \gamma^n}{\theta - \gamma^n}, \tag{2}$$

Also, Pel-Lucasl sequence PL_n is defined by

$$PL_n = 2PL_{n-1} + PL_n; \quad PL_0 = 2, PL_1 = 2. \tag{3}$$

Stakhov and Rozin[15], by applying concept of the classical hyperbolic functions, introduced new class of hyperbolic functions and stablished intresting results and theorems. In this paper, inspiration by the work of Stakhov and Rozin, we introduce new class of hyperbolic function based on the classical hyperbolic functions and deffinition of Pell and Pell-Lucas sequences, namely, symmetric Hyperbolic Pell sine function and symmetric hyperbolic Pell cosine function. These functions, combine the concept of classical hyperbolic functions and recursive Pell sequence and Pell-Lucas sequence. We study some properties of these new hyperbolic functions and present some identities about these functions.

2 Symmetric hyperbolic Pell sine function and Pell cosine function

In this section, inspiration by the work of Stakhov and Rozin[15], we shall introduce new calss of hyperbolic functions, namely, symmetric hyperbolic Pell sine function and hyperbolic Pell cosine function.

It is well known that the classical hyperboic functions are given by

$$sinh(x) = \frac{e^x - e^{-x}}{2}, \quad cosh(x) = \frac{e^x - e^{-x}}{2}. \tag{4}$$

Based on the definition of classical hyperbolic functions, we define new class of hyperboilc functions.

Definition 2.1. The symmetric hyperbolic Pell sine function and Pell cosine function denoted respectively by $Sph(x)$ and $Cph(x)$, are defined by

$$Sph(x) = \frac{\theta^x - \theta^{-x}}{2\sqrt{2}}, \quad Cph(x) = \frac{\theta^x + \theta^{-x}}{2\sqrt{2}}, \tag{5}$$

where $\theta = 1 + \sqrt{2}$.

Also, the symmetric hyperbolic Pell-Lucas sine function and symmetric hyperbolic Pell-Lucas cosine function denoted by $Splh(x)$ and $Cplh(x)$, are defined by

$$Splh(x) = \theta^x - \theta^{-x}, \quad Cplh(x) = \theta^x + \theta^{-x}. \tag{6}$$

Remark 2.2. Following relations, state the connection between classical hyperbbolic functions and symmetric hyperbolic Pell sine function and hyperbolic Pell cosine function.

$$Sph(x) = \frac{\theta^x - \theta^{-x}}{2\sqrt{2}} = (\frac{e^{x(ln\theta)} - e^{-x(ln\theta)}}{2})\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[sinh(xln\theta)\right],$$

and

$$Cph(x) = \frac{\theta^x + \theta^{-x}}{2\sqrt{2}} = (\frac{e^{x(ln\theta)} + e^{-x(ln\theta)}}{2})\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[cosh(x(ln\theta))\right].$$

Also, we obtain that

$$Sph(-x) = \frac{\theta^{-x} - \theta^{-(-x)}}{2\sqrt{2}} = \frac{\theta^{-x} - \theta^{x}}{2\sqrt{2}} = -[\frac{\theta^{x} - \theta^{-x}}{2\sqrt{2}}] = -Sph(x),$$

and

$$Cph(-x) = \frac{\theta^{-x} + \theta^{-(-x)}}{2\sqrt{2}} = \frac{\theta^{-x} + \theta^{x}}{2\sqrt{2}} = Cph(x).$$

Thus, the symmetric hyperbolic Pell sine function is an **odd** function and symmetric hyperbolic Pell cosine function is an **even** function.

3 Recursive properties of symmetric hyperbolic Pell sine function and Pell cosine function

In this section, according to definition of Pell sequence and Pell-Lucas sequence, we present some identities about these new class of hyperbolic functions. The following theorem states a comparison between Pell and PellLucas sequences and new hyperbolic functions.
We start this section by stating a lemma.

Lemma 3.1. *Following identities hold for $\theta = 1 + \sqrt{2}$.*
(a) $1 + 2\theta = \theta^2$,
(b) $\theta^{-2} = 1 - 2\theta^{-1}$,
(c) $\theta^{-1} + 2 = \theta$.

Proof. As we know, θ is the root of the charactristic equation $r^2 - 2r - 1 = 0$. So, these identities can be proved by direct calculation. □

Theorem 3.2. *Let $Sph(x)$ denotes the symmetric hyperbolic Pell sine function and $Cph(x)$ denotes the symmetric hyperbolic Pell cosine function. Then $Sph(x)$ and $Cph(x)$ satisfy following identities.*

$$(a) \quad Sph(x+2) = 2Cph(x+1) + Sph(x), \qquad (b) \quad CPh(x+2) = 2Sph(x+1) + Cph(x). \tag{7}$$

Proof. We prove (a).

$$2Cph(x+1) + Sph(x) = 2[\frac{\theta^{x+1} + \theta^{-x-1}}{2\sqrt{2}}] + \frac{\theta^x - \theta^{-x}}{2\sqrt{2}} = \frac{\theta^x(2\theta + 1) - \theta^{-x}(1 - 2\theta^{-1})}{2\sqrt{2}}.$$

Applying lemma(3.1), we obtain

$$Sph(x+2) = \frac{\theta^x\theta^2 - \theta^{-x}\theta^{-2}}{2\sqrt{2}} = \frac{\theta^{x+2} - \theta^{-x-2}}{2\sqrt{2}} = Sph(x+2).$$

This completes the proof. Part (b) can be proved by similar method. □

Similarly, one can prove follwing recursive identities of the symmetrc hyperbolic Pell-Lucas sine function and symmetric hyperbolic Pell-Lucas cosine function.

Theorem 3.3. *Let $Splh(x)$ denotes the symmetric hyperbolic Pell-Lucas sine function and $Cplh(x)$ denotes the symmetric hyperbolic Pell-Lucas cosine function. Then, symmetric hyperbolic functions $Splh(x)$ and $Cplh(x)$ satisfy following identities.*

$$Spl(x+2) = 2Cplh(x+1) + Splh(x), \qquad Cplh(x+2) = 2Splh(x+1) + Cplh(x). \tag{8}$$

Next theorem states two identities which are similar to the important identity $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$.

Theorem 3.4. *Following identities hold for symmetric hyperbolic functions $Sph(x)$ and $Cph(x)$.*

$$Cph^2(x) - Sph(x+1)\ Sph(x-1) = \frac{1}{8}[2 + Cph(2)], \tag{9}$$

$$Sph^2(x) - Cph(x+1)\ Cph(x-1) = -\frac{1}{8}[2 + Sph(2)]. \tag{10}$$

Proof. We prove first identity. Other identity similarly can be proved.

$$Cph^2(x) - Sph(x+1)\ Sph(x-1) = \left(\frac{\theta^x + \theta^{-x}}{2\sqrt{2}}\right)^2 - \left(\frac{\theta^{x+1} - \theta^{-x-1}}{2\sqrt{2}}\right)\left(\frac{\theta^{x-1} - \theta^{-x+1}}{2\sqrt{2}}\right)$$

$$= \frac{\theta^{2x} + \theta^{-2x} + 2}{(2\sqrt{2})^2} - \left[\frac{\theta^{2x} - \theta^2 - \theta^{-2} + \theta^{-2x}}{(2\sqrt{2})^2}\right] = \frac{2 + \theta^2 + \theta^{-2}}{(2\sqrt{2})^2} = \frac{1}{8}[2 + Cph(2)].$$

□

Corresponding to the last theorem, one can prove the following identities about symmetric hyperbolic Pell-Lucas sine and Pell-Lucas cosine functions.

Theorem 3.5. *Symmetric hyperbolic functions $Splh(x)$ and $Cplh(x)$ satisfy following identities.*

$$Cplh^2(x) - Splh(x+1)\ Splh(x-1) = 2 + Cplh(2), \tag{11}$$

$$Splh^2(x) - Cplh(x+1)\ Cplh(x-1) = -[2 + Splh(2)]. \tag{12}$$

Proof. This theorem can be proved by similar manner which we used for the last theorem.

□

4 Hyperbolic properties

In this section, we study some properties of symmetric hyperbolic Pell sine function and Pell cosine function that are similar to the properties of the classical hyperbolic functions.

Theorem 4.1. *Following identity is valid for the symmetric hyperbolic Pell sine and cosine functions.*

$$Cph^2(x) - Sph^2(x) = \frac{1}{2}. \tag{13}$$

Proof.

$$Cph^2(x) - Sph^2(x) = \left(\frac{\theta^x + \theta^{-x}}{2\sqrt{2}}\right)^2 - \left(\frac{\theta^x - \theta^{-x}}{2\sqrt{2}}\right)^2 = \frac{\theta^{2x} + \theta^{-2x} + 2 - \theta^{2x} - \theta^{-2x} + 2}{(2\sqrt{2})^2} = \frac{4}{8} = \frac{1}{2}.$$

□

Next theorem presents some equations that are similar to the classical hyperbolic equation $cosh(x+y) = cosh(x)\ cosh(y) + sinh(x)\ sinh(y)$ and $cosh(x-y) = cosh(x)\ cosh(y) - sinh(x)\ sinh(y)$.

Theorem 4.2. *Following eqations are valid for the symmetric hyperbolic Pell sine function and Pell cosine function.*

$$Cph(x)\ Cph(y) + Sph(x)\ Sph(y) = \frac{\sqrt{2}}{2}\ Cph(x+y), \tag{14}$$

and

$$Cph(x)\ Cph(y) - Sph(x)\ Sph(y) = \frac{\sqrt{2}}{2}\ Cph(x-y). \tag{15}$$

Proof. We prove the first identity.

$$Cph(x)\ Cph(y) + Sph(x)\ Sph(y) = \left(\frac{\theta^x + \theta^{-x}}{2\sqrt{2}}\right)\left(\frac{\theta^y + \theta^{-y}}{2\sqrt{2}}\right) + \left(\frac{\theta^x - \theta^{-x}}{2\sqrt{2}}\right)\left(\frac{\theta^y - \theta^{-y}}{2\sqrt{2}}\right)$$

$$\frac{\theta^{x+y} + \theta^{x-y} + \theta^{-x+y} + \theta^{-x-y} + (\theta^{x+y} - \theta^{x-y} - \theta^{-x+y} + \theta^{-x-y})}{(2\sqrt{2})^2} = \frac{2\theta^{x+y} + 2\theta^{-x-y}}{(2\sqrt{2})^2}$$

$$= \frac{\sqrt{2}}{2}\ Cph(x+y).$$

This completes the proof. Second identity similarly can be proved. □

Corollary 4.3. *If we set $y = x$ in the last theorem, we obtain the following identity.*

$$Cph^2(x) + Sph^2(x) = \frac{\sqrt{2}}{2}\ Cph(2x). \tag{16}$$

Now, we prove that similar identities hold for $Sph(x+y)$ and $Sph(x-y)$.

Theorem 4.4. *$Sph(x)$ and $Cph(y)$ satisfy the following equations.*

$$Sph(x)\ Cph(y) + Cph(x)\ Sph(y) = \frac{\sqrt{2}}{2}\ Sph(x+y), \tag{17}$$

and

$$Sph(x)\ Cph(y) - Cph(x)\ Sph(y) = \frac{\sqrt{2}}{2}\ Sph(x-y). \tag{18}$$

Proof. We prove the first identity.

$$Sph(x)\ Cph(y) + Cph(x)\ Sph(y) = \left(\frac{\theta^x - \theta^{-x}}{2\sqrt{2}}\right)\left(\frac{\theta^y + \theta^{-y}}{2\sqrt{2}}\right) + \left(\frac{\theta^x + \theta^{-x}}{2\sqrt{2}}\right)\left(\frac{\theta^y - \theta^{-y}}{2\sqrt{2}}\right)$$

$$\frac{\theta^{x+y} + \theta^{x-y} - \theta^{-x+y} - \theta^{-x-y} + (\theta^{x+y} - \theta^{x-y} + \theta^{-x+y} - \theta^{-x-y})}{(2\sqrt{2})^2} = \frac{2\theta^{x+y} - 2\theta^{-x-y}}{(2\sqrt{2})^2}$$

$$= \frac{\sqrt{2}}{2}\ Sph(x+y).$$

This completes the proof. Second identity similarly can be proved. □

Corollary 4.5. *If we set $y = x$ in the last theorem, we obtain the following identity.*

$$Sph(2x) = 2\sqrt{2}\ Sph(x)\ Cph(x). \tag{19}$$

5 Conclusion

In this paper, we introduced new class of hyperbolic functions based on the classical hyperbolic functions and properties of Pell and Pell-Lucas sequences, namely, symmetric Hyperbolic Pell sine function and symmetric hyperbolic Pell cosine function. These functions, unite the concept of classical hyperbolic functions and recursive Pell sequence and Pell-Lucas sequence. We study some properties of these new hyperbolic functions and present some identities about these functions. One can combine the concept of classical hyperbolic funtions to other well-known number sequences and get new results about them for the future works.

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