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New Results of t^{ω}-Normed Approach Space

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ABSTRACT

In this paper, a new convergence formula, cluster point , t^{ω}-Cauchy sequence, t^{ω}-convergent, t^{ω}-completeness , and sequentially contraction in t^{ω}-approach space are defined. In addition, the contraction condition is proved to be essential and important to obtain the sequential contraction function. Also, a new structure for the norm in t^{ω}- approach space is put, which is called t^{ω}-approach – Banach space. t^{ω}- normed approach space with uniform condition are introduced to be a Hausdorff space as well as t^{ω}-normed approach space is shown a complete if and only if the metric space generated from t^{ω}-approach space is a complete, so, every finite – dimensional t^{ω}-approach normed space is proven to be t^{ω}-complete. Some results and properties in this field are introduced and discussed .

KEYWORDS: t^{ω}-approach space, t^{ω}-contraction, t^{ω}-approach- metric space, t^{ω}-approach normed , t^{ω}-approach – Banach space.

1 INTRODUCTION

The distance between points and sets in a metric space was studied by R.Lowen [4] in (1989). The measure of Lindelof and separability in approach space was studied in (1994) by R. Baekeland and R. Lowen [7]. In (1996), the development of the fundamental theory of approximation was studied by R. Lowen [13]. In (1999), R.Lowen, Y. Jinlee [2] introduced and defined the notions of approach Cauchy structure and ultra-approach Cauchy structure. In (2000) and (2003) R. Lowen and M. Sioen [8,10] defined the important definitions of some separation axioms in the approach spaces and found the relationship between them.

In (2000), R. Lowen and B. Windels [14] defined the essential notions of an approach groups spaces, semi-group spaces, and uniformly convergent. In(2003), a complete theory for all approach spaces with an underlying topology which agrees with the usual metric completion theory for metric spaces was defined by R. Lowen, M. Sion and D. Vaughan [3]. In (2004), an approach vector spaces was studied by R. Lowen and S. Verwuwlgen [5].

In (2004), R. Lowen, C. Van Olmen, and T. Vroegrijk [9] found very important relationship between Functional ideas and Topological Theories. In (2006), G. C. L. Brümmer, M, Sion [16] studied and developed abicompletion theory for the category of approach spaces in sense of Lowen [20] which is extended the completion theory obtained in [14].

In (2009), J.Martnez-Moreno, A. Roldan and C. Roldan [17] introduced the necessary notion of fuzzy approach spaces generalization of fuzzy metric spaces and proved some properties of fuzzy approach spaces. In (2009), some notions and relations in approach Theory were discussed by R. Lowen and C.Van Olmen [11].

In(2013), G. Gutierres. D. Hofmann [12] introduced and studied the notion of cocompleteness for an approach spaces so, proved some properties in cocompleteness approach space. In (2013), K.Van Opdenbosch [18] introduced a new isomorphic characterizations of approach spaces, pre-approach spaces, convergence approach spaces, uniform gauge spaces, topological spaces , convergence spaces, topological spaces, metric spaces, and uniform spaces.

In (2014), R.Lowen, S.Sagiroglu [22] defined the possibility to weak the notion of an approach spaces to incorporate not only topological and metric spaces but also closure spaces. In (2015), R.Lowen [6] discussed two new types of numerically structured spaces which are required: approach spaces on the local level and uniform gauge spaces on the uniform level.

In (2016), several generalizations of known theorems of fixed point, and theorems for common fixed points of mapping to 2- Banach space were proved by R. Malčeski, A.Ibrahimi [21]. E.Colebunders, M. Sion[1] solved and prove some important consequences on real-valued contractions in (2017).

In (2017) and (2019), M. Baran and M. Qasim [20,22] characterized the local distance-approach spaces, approach spaces, and gauge-approach spaces so, compared them with usual approach spaces. In (2018), W. Li, Dexue Zhang [21] defined the Smyth complete. In this paper, we define t^{ω}- normed approach space, and t^{ω}-Banach approach space structure on χ so, obtain their properties. Quantitative results are proved, which imply their classical qualitative counterparts.

These results provide an introduction to t^{ω}- Banach approach spaces, which are t^{ω}-complete normed vector approach spaces. This work is also studied the category-theoretic properties of t^{ω}- Banach approach spaces. The extension t^{ω}- Banach approach space by t^{ω}- complete normed approach space is also introduced. This extension leads to expand the space of the norm though. In addition, we make an additional condition on the norm structure , that is $t^{\omega}_{\|.\|}(\{x\}, A) := sup_{x\in\chi} \inf_{a\in A} \|x - a\|$. This means the distance generated by norm function between tow subsets of power set in t^{ω}-approach space. In this case, the conditions and functions have been fulfilled, we want to find any Cauchy sequence convergent in t^{ω}-approach space and the space has become t^{ω}- Banach approach space. Some properties of t^{ω}-Banach space are proved.

The main aim of this work is to find and to discuss new results in convergent sequences in t^{ω}-approach spaces. We prove t^{ω}-approach space is complete if and only if $(\chi, d_{t^{\omega}})$ is complete, we also introduce sequentially contraction and show that it is equivalent contraction. t^{ω}-normed approach space is defined, and we prove some results such that every t^{ω}-uniform normed approach space $(\chi, \|.\|, {t^{\omega}}_{\|.\|})$ is a Hausdorff, and if a sequence in t^{ω}-normed approach convergent sequence, then the sequence is bounded. Also, some new results in t^{ω}-normed approach space and t^{ω}-Banach approach space are studied. Futhermore, in this paper, a new definition of t^{ω}-convergent of cluster point in t^{ω}-approach space, t^{ω}-Cauchy sequence, t^{ω}-convergent and t^{ω}-complete are discussed. We prove that a function $\varrho: (\chi, t^{\omega}) \to (Y, t^{\omega'})$ between t^{ω}-approach spaces is a contraction if and only if it is sequentially contraction, we also discussed every finite –dimensional t^{ω}-normed approach space is complete and consequent t^{ω}-Banach approach space.

Also, the t^{ω}-metric approach space $(\chi, d_{t^{\omega}})$ is not to be t^{ω}-normed approach space. t^{ω}- normed approach space $(\chi, \|.\|, {t^{\omega}}_{\|.\|})$ is t^{ω}-complete if and only if t^{ω}- metric approach space $(E, d_{\|.\|})$ is complete by means every t^{ω}-uniform normed approach space $(\chi, \|.\|, {t^{\omega}}_{\|.\|})$ is a Hausdorff space. If $(\chi, \|.\|, {t^{\omega}}_{\|.\|})$ is t^{ω}-normed approach space, and $\{x_n\}$ is a convergent sequence in χ, then, the sequence $\{x_n\}$ in χ is norm bounded.

New results in t^{ω}- normed approach space and convergent are given. This work is divided into six sections: Section one shows the introduction of the research. In section two, preliminaries with basic definitions are introduced. In section three, new results in t^{ω}-convergent sequences in t^{ω}- approach spaces are studied.

We also explain the relationship complete and t^{ω}- complete in t^{ω}- approach space, so we introduce the definition of t^{ω}- normed approach space and prove some results in t^{ω}- normed approach space, and some results of t^{ω}- normed approach space are given. Section for, we discuss and explain the important and the essential conclusions of the research.

2 PAPER FORMAT

We start this work by definition of central notion, namely t^{ω}-approach distance.

Definition 2.1[23]: Let χ be a non-empty set. A collection $(t^{\omega})_{\omega<\infty}$ of functions t^{ω} : $2^{\chi} \times 2^{\chi} \rightarrow [0,\infty]$ is known as t^{ω}-approach distance on χ if this function satisfies the following properties:

(t_1) $\forall\omega \in \mathbb{R}^+, \forall A, B \in 2^{\chi}: t^{\omega}(A,B) = 0 \Rightarrow A = B$,

(t_2) $\forall\omega \in \mathbb{R}^+, \forall A \in 2^{\chi}$, then $t^{\omega}(A,\emptyset) = \infty$,

(t_3) $\forall\omega \in \mathbb{R}^+, \forall A, B, C \in 2^{\chi}: t^{\omega}(A, B \cap C) = max\{t^{\omega}(A,B), t^{\omega}(A,C)\}$,

(t_4) $\forall\omega \in \mathbb{R}^+, \forall A, B \in 2^{\chi}, \forall\varepsilon < \omega: t^{\omega}\left(A^{\omega}_{(\varepsilon)}, B\right) \leq t^{\omega}(A,B) + \varepsilon$ where

$A^{\omega}_{(\varepsilon)} = \{x \in \chi | t^{\omega}(\{x\}, B) \leq \omega - \varepsilon\}$.

A pair (χ, t^{ω}) where the function t^{ω} is a distance and this pair is called t^{ω}- approach space and denoted by t^{ω}-app-space.

Definition 2.2: Let (χ, t^{ω}) be t^{ω}- app-space, a sequence $\{x_n\}$ in the t^{ω}-app- space is the convergent sequence to $x \in \chi$ if $lim_{n\rightarrow\infty} \inf_{x\in A} t^{\omega}(\{x_n\}, A) = 0$ and $lim_{n\rightarrow\infty} sup_{x\in A} t^{\omega}(\{x_n\}, A) = 0$.

Definition 2.3[23]: Let (χ, t^{ω}) and $(Y, t^{\omega'})$ are app- spaces. A

function $\varrho: \chi \rightarrow Y$ is known as $t^{\omega}-$ contraction if for all $\omega < \infty$, and for all $A, B \in 2^{\chi}$, $t^{\omega'}(\varrho((A), \varrho(B)) \leq k\, t^{\omega}(A,B)$, for some $k \in [0,1]$.

Definition 2.4 : Let (χ, t^{ω}) and $(Y, t^{\omega'})$ are app-spaces. A function

$\sigma: \chi \rightarrow Y$ is called sequentially contraction if $lim_{n\rightarrow\infty} t^{\omega'}\left(\sigma(\{x_n\}), \sigma(A)\right) = 0$ whenever $lim_{n\rightarrow\infty} t^{\omega}(\{x_n\}, A) = 0$.

Theorem 2.5 : Let (χ, t^{ω}) and $(Y, t^{\omega'})$ are t^{ω}-app-spaces , then a function

$\varrho: \chi \rightarrow Y$ is $t^{\omega}-$contraction if and only if ϱ sequentially contraction.

Proof: Suppose ϱ is $t^{\omega}-$contraction, and $\{x_n\}$ is a convergent sequence in χ, such

that $lim_{n\rightarrow\infty} sup_{x\in A} t^{\omega}(\{x_n\}, A) = 0$ and $lim_{n\rightarrow\infty} \inf_{x\in A} t^{\omega}(\{x_n\}, A) = 0$, since ϱ is $t^{\omega}-$contraction, then

$t^{\omega'}\left(\varrho(\{x_n\}), \varrho(A)\right) \leq k t^{\omega}(\{x_n\}, A) = 0$.

Thus, $lim_{n\to\infty}sup_{x\in A}t^{\omega'}(\varrho(\{x_n\}),\varrho(A)) \leq klim_{n\to\infty}sup_{x\in A}t^{\omega}(\{x_n\},A) = 0$,

$lim_{n\to\infty}\inf_{x\in A}t^{\omega}(\{x_n\},A) = 0$, and $lim_{n\to\infty}sup_{x\in A}t^{\omega}(\{x_n\},A) = 0$.

$lim_{n\to\infty}\inf_{x\in A}t^{\omega}(\{x_n\},A) \geq lim_{n\to\infty}\inf_{x\in A}t^{\omega'}(\varrho(\{x_n\}),\varrho(A)) \geq lim_{n\to\infty}sup_{x\in A}t^{\omega'}(\varrho(\{x_n\}),\varrho(A))$

$\leq lim_{n\to\infty}sup_{x\in A}t^{\omega}(\{x_n\},A) = 0$.

$lim_{n\to\infty}sup_{x\in A}t^{\omega'}(\varrho(\{x_n\}),\varrho(A)) = 0$ and $lim_{n\to\infty}sup_{x\in A}t^{\omega}(\{x_n\},A) = 0$.

Therefore ϱ sequentially contraction.

Conversely, let ϱ is sequentially contraction

Suppose ϱ is not $t^{\omega}-$contraction,

Then $t^{\omega'}\big(\varrho(\{x\}),\varrho(A)\big) > t^{\omega}(\{x\},A)$.

$t^{\omega'}(\varrho(\{x_n\}),\varrho(A)) > t^{\omega}(\{x_n\},A)$.Suppose $\{x_n\}$ be a converge sequence in X,since ϱ is sequentially contraction, $lim_{n\to\infty}sup_{x\in A}t^{\omega'}(\varrho(\{x_n\}),\varrho(A)) = lim_{n\to\infty}\inf_{x\in A}t^{\omega'}(\varrho(\{x_n\}),\varrho(A)) = 0$.

Whenever $lim_{n\to\infty}\inf_{x\in A}t^{\omega}(\{x_n\},A) = 0$ and $lim_{n\to\infty}sup_{x\in A}t^{\omega}(\{x\},A) = 0$,

this impossible . ■

Definition 2.6[23]: A triple $(\chi,t^{\omega},*)$ is known as t^{ω}- app- semi-group if and only if

1. (χ,t^{ω}) is t^{ω}- app- space.
2. $(\chi,*)$ is a semi-group.
3. $*:\chi \otimes \chi \to \chi: (x,y) \mapsto x * y$ is a $t^{\omega}-$contraction.

Definition 2.7[23]: The triple $(\chi,t^{\omega},*)$ is known as t^{ω}- app- group if satisfy the following:

a) (χ,t^{ω}) is t^{ω}- app- space.

b) $(\chi,*)$ is a group.

c) $*:\chi \otimes \chi \to \chi: (x,y) \mapsto x + y$ is the t^{ω} –contraction .

d) $-:\chi \to \chi: x \to -x$ is $t^{\omega}-$contraction.

Definition 2.8[23]: Let F be a field and let χ be a non-empty power set with two binary operations :an addition and a scalar multiplication, $\forall\omega < \infty, t^{\omega}$ is an app- distance on 2^{χ}, then, a quadruple $(\chi,t^{\omega},+,.)$ is said to be t^{ω}- vector approach space if satisfy the following:

1) $(\chi,t^{\omega},+)$ is t^{ω}- app-group.

2) $(\chi,t^{\omega},.)$ is t^{ω}- app-semi group.

3) $\mu.(x+y) = \mu.x + \mu.y$, for all $\mu \in F$, for all $x,y \in \chi$.

4) $(x+y).\mu = x.\mu + y.\mu$, for all $\mu \in F$, for all $x,y \in \chi$.

5) $(\mu.\vartheta).x = \mu(\vartheta x)$, for all $x \in \chi$, for all $\mu,\vartheta \in F$.

6) $1. x = x$, for all $x \in \chi$.

Definition 2.9 : Let V and W be two t^{ω}-vector app- spaces on app- space over the same field F. A mapping $\varphi: V \rightarrow W$ is said to be (approach linear transformation) if the following hold:

1) $\varphi(a+b) = \varphi(a) * \varphi(b), a, b \in V$;

2) $\varphi(\lambda a) = \lambda\varphi(a)$ for all $\lambda \in F$, for all $a, b \in V$.

Definition 2.10 : Let $\varphi: V \rightarrow W$ be t^{ω}-approach linear map then the set

$ker\,(\varphi) = \{a \in V: \varphi(a) = O_W\} = \varphi^{-1}(O_W)$ is called the (t^{ω}-approach kernel of φ), where O_W is the t^{ω}-approach identity of W.

Theorem 2.11: Let $(V, T_V, t^{\omega})$and $(W, T_W, t^{\omega'})$ be t^{ω}- topological vector app- spaces, and the t^{ω}-approach linear map $\varphi: V \rightarrow W$ is $t^{\omega}-$contraction ,then the $ker(\varphi)$ is closed.

Proof: Suppose φ is the t^{ω} $-$contraction To prove $ker(\varphi)$ is closed set, let $\{x_n\}$ be a sequence convergent to x in $ker(\varphi)$ such that $lim_{n\rightarrow\infty}\inf_{\{x_n\}\in ker\,(\varphi)} t^{\omega}(\{x_n\}, A) = 0$ and $lim_{n\rightarrow\infty} sup_{\{x_n\}\in ker\,(\varphi)} t^{\omega}(\{x_n\}, A) = 0$, since φ is t^{ω} $-$contraction, that is $t^{\omega'}(\varphi(\{x_n\}), \varphi(A)) \leq k\, t^{\omega}(\{x_n\}, A)$.

Then, $0 = k lim_{n\rightarrow\infty}\inf_{A\in 2^V} t^{\omega}(\{x_n\}, A) \leq lim_{n\rightarrow\infty}\inf_{A\in 2^V} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big) \leq lim_{n\rightarrow\infty} sup_{A\in 2^V} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big) \leq k\, lim_{n\rightarrow\infty} sup_{A\in 2^V} t^{\omega}(\{x_n\}, A) = 0$,

$lim_{n\rightarrow\infty} sup_{A\in 2^V} t^{\omega'}(\varphi(\{x_n\}), \varphi(A)) = 0$ and $lim_{n\rightarrow\infty}\inf_{A\in 2^V} t^{\omega'}(\varphi(\{x_n\}), \varphi(A)) = 0$.

$\varphi(\{x_n\}) = 0$, so, $\varphi(\{x_n\}) = \{0_W\}$, $lim_{n\rightarrow\infty} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big) = 0$.

Then, $\varphi(\{x\}) = \{0\}$, that is $\{x\} \in ker(\varphi)$.

Conversely, suppose $ker(\varphi)$ is closed set,

Let $\{x_n\}$ be a sequence convergent to x in $ker(\varphi)$.

to prove $\varphi(\{x_n\})$ convergent to $\varphi(\{x\})$, since $ker(\varphi)$is closed, that is $\{x\} \in ker(\varphi)$

Assume $\varphi(\{x_n\})$ is not convergent to $\varphi(\{x_0\})$ in A that is φ not t^{ω} $-contraction$.

Then, $lim_{n\rightarrow\infty} sup_{x\in A} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big) \neq 0$ or $0 \neq lim_{n\rightarrow\infty}\inf_{x\in A} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big)$.

If $lim_{n\rightarrow\infty} sup_{x\in A} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big) \neq 0$ or $0 = lim_{n\rightarrow\infty}\inf_{x\in A} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big)$.

$0 = lim_{n\rightarrow\infty}\inf_{x\in A} t^{\omega}(\{x_n\}, A) > lim_{n\rightarrow\infty}\inf_{x\in A} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big)$, this means $t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big) < 0$.

This impossible.

If $lim_{n\rightarrow\infty} sup_{x\in A} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big) = 0$ or $0 \neq lim_{n\rightarrow\infty}\inf_{x\in A} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big)$.

$0 = lim_{n\to\infty} sup_{x\in A} t^{\omega}(\{x_n\}, A) < lim_{n\to\infty} sup_{x\in A} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big) = 0$.

Impossible.

If $lim_{n\to\infty} sup_{x\in A} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big) \neq 0$ and $0 \neq lim_{n\to\infty} \inf_{x\in A} t^{\omega'}\big(\varphi(\{x_n\}), \varphi(A)\big)$, but $\varphi(\{x_n\}) \in ker(\varphi)$.

Then, $lim_{n\to\infty} sup_{x\in A} t^{\omega'}\big(\varphi\{0\}, \varphi(A)\big) \neq 0$ and $0 \neq lim_{n\to\infty} \inf_{x\in A} t^{\omega'}\big(\varphi\{0\}, \varphi(\{x\})\big)$, that is $t^{\omega'}\big(\varphi\{0\}, \varphi(\{x\})\big) \neq 0, \varphi(\{x\}) \neq 0$, therefore $\{x\} \notin ker(\varphi)$, this impossible.

Thus, φ is sequentially contraction , then φ is t^{ω} −contraction. ■

3 Some Properties of Convergent Sequences in t^{ω} -Approach Spaces and t^{ω}-Normed Approach Space.

In this section, we introduce a new definition of convergence in t^{ω} - app- space and discuss many properties related to this convergence and put a basic definition of normed space.

Definition 3.1 : A set $A \in 2^{\chi}$ is said to be cluster point in t^{ω}- app- space (χ, t^{ω}) if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in χ ,such that, $\inf_{x\in A} t^{\omega}(\{x_n\}, A) = 0$ which is written by $\{x_n\}_{n=1}^{\infty} \to$ A,

we denoted the set of all cluster point in t^{ω} -approach space by $\Gamma(\chi)$.

Definition 3.2 : A sequence $\{x_n\}_{n=1}^{\infty}$ $in\ \chi$ is said to be Cauchy sequence in t^{ω} - app- space (t^{ω} – Cauchy sequence) if for every cluster point A,

$lim_{n\to\infty} \inf_{x_n\in A} t^{\omega}(\{x_n\}, A) = 0$, a sequence $\{x_n\}_{n=1}^{\infty}$ $in\ \chi$ is said to be t^{ω} −convergent sequence in t^{ω} -app-space if : $\exists x \in \chi, \exists \{x\} \in 2^{\chi}, \forall A \in \Gamma(\chi), : t^{\omega}(\{x_n\}, A) = 0$.

Proposition 3.3: Let (χ, t^{ω}) be t^{ω} - app- space, then the following are equivalent:

1- t^{ω} −Convergent sequence in t^{ω}-app-space.

2- $lim_{n\to\infty} \inf_{x\in A} t^{\omega}(\{x_n\}, A) = 0$ and $lim_{n\to\infty} sup_{x\in A} t^{\omega}(\{x_n\}, A) = 0$.

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be t^{ω} −convergent sequence in app-space. $\exists x \in \chi, \{x\} \in 2^{\chi}, \forall A \in \Gamma(\chi) : t^{\omega}(\{x_n\}, A) = 0. \forall A \in \Gamma(\chi) : \inf_{x\in A} t^{\omega}(\{x_n\}, A) = 0$ and $sup_{x\in A} t^{\omega}(\{x_n\}, A) = 0, \forall A \in \Gamma(\chi), lim_{n\to\infty} \inf_{x\in A} t^{\omega}(\{x_n\}, A) = 0$ and $lim_{n\to\infty} sup_{x\in A} t^{\omega}(\{x_n\}, A) = 0$.

Conversely, suppose the condition (2) is true.

$lim_{n\to\infty} \inf_{x\in A} t^{\omega}(\{x_n\}, A) = 0$ and $lim_{n\to\infty} sup_{x\in A} t^{\omega}(\{x_n\}, A) = 0$.

Then, A is cluster point, that is $\inf_{x\in A} t^{\omega}(\{x_n\}, A) = 0 . \exists x \in \chi, \{x\} \in 2^{\chi}, \forall A \in \Gamma(\chi), : t^{\omega}(\{x_n\}, A) = 0$. Thus, $\{x_n\}_{n=1}^{\infty}$ is t^{ω} −convergent sequence in t^{ω} -app-space.

Remark 3.4: Every t^{ω} −convergent sequence is t^{ω} −Cauchy sequence.

Proposition 3.5 : If χ is t^{ω} - app- space the following are equivalent;

(1) $\{x_n\}_{n=1}^{\infty}$ is a t^{ω} −convergent sequence in t^{ω} app-space.

(2) $sup_{A\in\Gamma(\chi)}\inf_{x\in A}d_{t^{\omega}}(x_n, x) = 0$.

Proof: It is clear.

Proposition 3.6: If $(\chi, t^{\omega}{}_{d})$ is t^{ω} - metric app- space and $\{x_n\}_{n=1}^{\infty}$ be Sequence in χ, then it is t^{ω} -Cauchy sequence in (χ, d) if and only if it is $t^{\omega}-$ Cauchy sequence in $(\chi, t^{\omega}{}_{d})$.

Proof: Firstly, let $\{x_n\}_{n=1}^{\infty}$ be a t^{ω} −Cauchy sequence in $(\chi, d_{t^{\omega}})$,then we have that $\inf_{x_n\in A}t^{\omega}(\{x_n\}, A) = 0.$ This implies that $\inf_{x_n\in A}t^{\omega}(\{x_n\}, \{x_m\}) = \inf_{x_n\in A}\inf_{x_m\in A}d(x_n, x_m) = 0.$

That is $d(x_n, x_m) = 0,$ then $\{x_n\}_{n=1}^{\infty}$ is left Cuachy sequence.

Also,$\inf_{x_n\in A}t^{\omega}(\{x_m\}, \{x_n\}) = \inf_{x_n\in A}\inf_{x_m\in A}d(x_m, x_n) = 0,$ that is $d(x_m, x_n) = 0, \{x_n\}_{n=1}^{\infty}$ is right Cauchy sequence. Thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in (χ, d).

Conversely, if $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in (χ, d).

Then it is left and right Cauchy sequence, for all $\varepsilon > 0$, there exists $N \in Z^{+}$

such that $d(x_m, x_n) < \varepsilon,$ for all $m, n \geq N, m \leq n,$ and for all $\varepsilon > 0,$ there exists $N \in Z^{+}$,such that $d(x_n, x_m) < \varepsilon$, for all $m, n \geq N, n \leq m.$ $\inf_{x_n\in A}t^{\omega}(\{x_n\}, A) = \inf_{x_m\in A}\inf_{x_n\in X}d(x_m, x_n) = 0.$

Hence $\{x_n\}_{n=1}^{\infty}$ is t^{ω} −Cauchy sequence in app- space.

Definition 3.7: t^{ω}- approach space is called $t^{\omega}-$complete if every $t^{\omega}-$Cauchy is $t^{\omega}-$convergent in (χ, t^{ω}).

Theorem 3.8: t^{ω} -approach space (χ, t^{ω}) is t^{ω} −complete space if and only if $(\chi, d_{t^{\omega}})$ is complete.

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be a Caushy sequence in (χ, d), then it is t^{ω} −Cauchy sequence in (χ, t^{ω}) by proposition (3), since (χ, t^{ω}) is t^{ω} -complete, there exists $x \in A$, for all $A \in \Gamma(A)$ such that $t^{\omega}(\{x_n\}, A) = 0, sup_{A\in\Gamma(A)}\inf_{x\in A}d_{t^{\omega}}(x_n, x) = 0.$

Then $d_{t^{\omega}}(x_n, x) = 0,$ that is $(\chi, d_{t^{\omega}})$ is complete.

Conversely, Let $\{x_n\}_{n=1}^{\infty}$ be $t^{\omega}-$Cauchy sequence in $(\chi, d_{t^{\omega}})$.

The sequence $\{x_n\}$ is left and right sequence in $(\chi, d_{t^{\omega}}).(\chi, d)$ is complete, that is $lim_{n\to\infty}d_{t^{\omega}}(x_n, x) = 0,$ that is $lim_{n\to\infty}\inf_{x\in A}d_{t^{\omega}}(x_n, x) = 0,$ and $lim_{n\to\infty}\inf_{x\in A}d_{t^{\omega}}(x_n, x) =$
$0, t^{\omega}(\{x_n\}, A) = sup_{A\in\Gamma(\chi)}\inf_{x\in A}d_{t^{\omega}}(x_n, x) = 0,$ that is $\exists x \in \chi, \forall A \in \Gamma(\chi) : t^{\omega}(\{x_n\}, A) = 0.$

Thus, $\{x_n\}_{n=1}^{\infty}$ is convergent in t^{ω}-approach space (χ, t^{ω}). ■

Definition1 3.9: A subspace of a product of t^{ω}-metric approach spaces is called a t^{ω}-uniform approach space that is t^{ω}-uniform distance.

Definition 3.10 : Let χ be t^{ω}- vector app- space. A triple $(\chi, \|.\|, {t^{\omega}}_{\|.\|})$ said to be t^{ω}-normed approach space if satisfy the following :

(1) $\|x\| = 0$ if and only if $x = 0$, for all $x \in X$.

(2) $\|\lambda.x\| = |\lambda|.\|x\| \; \forall \lambda \in F, x \in X$

(3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$.

(4) $\|x\| \geq 0, \forall x \in X$

(5) ${t^{\omega}}_{\|.\|}(\{x\}, A) = sup_{x \in X} \inf_{a \in A} \|x - a\|$.

Remark 3.11 :

1. Every t^{ω}-normed approach space is normed space.

2. t^{ω}-normed space is not necessary t^{ω}- normed approach space. The following example shows that:

Example 3.12: Let $C\,[-1, 1]$ be a set of all continuous functional on $[-1, 1]$, a vector space $C\,[-1, 1]$ is normed space under the norm define:

By $\|f\| = sup_{x \in [-1,1]}\{|f(x)|\}$, when $f(x) = x - 1$ for all $x \in X$.

But that is not t^{ω}-normed vector app-space because:

Since condition: for $A = \{-1, 0, 1\}$

$d_{{t^{\omega}}_{\|.\|}}(x, y) = sup_{x \in X} \inf_{a \in A} \|f(x) - f(a)\| = 1.$

$But\; d_{\|.\|}(x, y) = \|x - a\| = sup_{x \in [-1,1]}\{|f(x) - f(a)|\} = 2.$

Definition1 3.13 : t^{ω}-Banach approach space is $t^{\omega}-$ complete normed approach space.

Proposition 3.14 : Every finite –dimensional t^{ω}- normed app- space is $t^{\omega}-$complete and consequent t^{ω}-Banach app- space.

Proof: Assume $dim\,(\chi) = n > 0, \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is t^{ω}-app-basis of χ.

Let $\{x_m\}_{m=1}^{n}$ be a t^{ω} −Cauchy sequence in χ,

$lim_{n \to \infty} \inf_{x_m \in A} t^{\omega}(\{x_m\}, A) = 0$.

For, $x_m = \sum_{i=1}^{n} \alpha_{im}\varphi_i$, $y_l = \sum_{i=1}^{n} \alpha_{il}\varphi_i$,

$$0 = lim_{n\to\infty}\inf_{y_l\in A} t^{\omega}(\{x_m\}, A) = lim_{n\to\infty}\inf_{y_l\in A}\inf_{y\in A} d_{t^{\omega}_{\|.\|}}(x_m, y)$$

$$= lim_{n\to\infty}\inf_{y_l\in A}\inf_{y\in A} d_{t^{\omega}_{\|.\|}}(x_m, y_l)$$

$$= lim_{n\to\infty}\inf_{y_l\in A}\inf_{y\in A} ||x_m - y_l|| \text{ ; that is } \sum_{i=1}^{n} ||\alpha_{im} - \alpha_{il}|| = 0.$$

Then $\{\alpha_{im}\}$ is t^{ω}-Cauchy sequence in real field $\mathbb{R}$ or complex field $\mathbb{C}$, since real field $\mathbb{R}$ or complex field $\mathbb{C}$ are complete, therefore ; for all i there exists $a_i \in F$ such that $lim_{n\to\infty}\alpha_{im} = a_i, put\ x = \sum_{i=1}^{n} \alpha_i\varphi_i$.

There exists $x \in A$, for all $A \in 2^{\chi}$, $lim_{n\to\infty}\inf_{x\in A} t^{\omega}(\{x_m\}, A) = 0$

Thus χ is t^{ω} −complete. ■

Let this follows from the fact that both $\mathbb{R}$ and $\mathbb{C}$ are complete and from the fact that every finite – dimentional is isomorphism to $\mathbb{R}^n$ or $\mathbb{C}^n$ for some n.

Proposition 3.15 : If E_1 and E_2 are t^{ω}- normed vector app- spaces, and f: $E_1 \to E_2$ is a surjective linear function, then the following properties are equivalent:

(1) f: $(E_1 , \|.\|_1, t_1^{\omega}) \to (E_2 , \|.\|_2, t_2^{\omega})$ is t^{ω}−contraction.

(2) (E_2 , t_2^{ω}) is t_2^{ω} − complete space whenever (E_1 , t_1^{ω}) is t_1^{ω} − complete.

Proof: If f: $E_1 \to E_2$ is t^{ω}−contraction.

Then for every $x \in E_1$ and each subset $A \subset E_1$.

$t_2^{\omega}\big(\text{f}(\{x\}), \text{f}(A)\big) \le k t_1^{\omega}(\{x\}, A)$.

If $(E_1 , \|.\|_1)$ is t^{ω}-Banach app-space.

To prove (E_2 , t_2^{ω}) is t_2^{ω}− complete space.

Let $\{y_n\}$ be a t_2^{ω}−Cauchy sequence in E_2 , then there exists $\{x_n\}$ such that f($\{x_n\}$) = $\{y_n\}$.

$lim_{n\to\infty} \inf_{x_n\in A} t_2^{\omega} (\{x_n\}, A) = 0$, then $lim_{n\to\infty} \inf_{x_n\in B} t_2^{\omega}\big(\text{f}(\{x_n\}), \text{f}(B)\big) = 0$, where $\text{f}(B) = A$. Since f $is\ t^{\omega}$ − contraction,

$0 = lim_{n\to\infty} \inf_{x_n\in B} t_2^{\omega}\big(\text{f}(\{x_n\}), \text{f}(B)\big) > k lim_{n\to\infty} \inf_{x_n\in A} t_1^{\omega}(\{x_n\}, A)$, for some $k \in [0,1]$

Hence, $k lim_{n\to\infty} \inf_{x_n\in A} t_1^{\omega}(\{x_n\}, A) = 0$, for some $k \in [0,1]$

That is $\{x_n\}$ $is\ t^{\omega}$ − Caushy sequence in E_1 , E_2 $is\ t^{\omega}$ − complete app − space.

There exists $x \in B$, for all $B \subseteq E_1$, such that $lim_{n\to\infty} \inf_{x\in A} t_1^{\omega}(\{x_n\}, B) = 0$

$t_2^{\omega}\big(\text{f}(\{x_n\}), \text{f}(B)\big) \le k t_1^{\omega}(\{x_n\}, B)$= 0, for some $k \in [0,1]$

$k lim_{n\to\infty} \inf_{x\in A} t_1^{\omega}(\{x_n\}, A)$= 0 and $k lim_{n\to\infty}\ sup_{x\in A}\ t_1^{\omega}(\{x_n\}, A)$= 0

$lim_{n\to\infty}\ sup_{x\in A}\ t_2^{\omega}(\text{f}(\{x_n\}), \text{f}(A)) \le k lim_{n\to\infty}\ sup_{x\in A}\ t_1^{\omega}(\{x_n\}, A)$= 0

$lim_{n\to\infty}$ $\inf_{x\in A}$ $t_2^{\omega}(f(\{x_n\}), f(A)) \geq k lim_{n\to\infty}$ $\inf_{x\in A}$ $t_1^{\omega}(\{x_n\}, A)$= 0, for some $k \in [0,1]$

$lim_{n\to\infty}$ $sup_{x\in A}$ $t_2^{\omega}(f(\{x_n\}), f(B)) \leq 0$.

Then $lim_{n\to\infty}$ $\inf_{x\in E_2}$ $t_2^{\omega}(f(\{x_n\}), f(A))$= 0 and $lim_{n\to\infty}$ $sup_{x\in E_2}$ $t_2^{\omega}(f(\{x_n\}), f(A))$ = 0 .

Then (E_2, t_2^{ω}) is $t_2^{\omega}-$complete app-space.

Conversely, suppose f is not $t^{\omega}-$contraction $t_2^{\omega}\big(f(\{x_n\}), f(B)\big) > k t_1^{\omega}(\{x_n\}, B)$, for some $k \in [0,1]$.

Let $\{x_n\}$ be a $t^{\omega}-$convergent sequence in E_1 , that is $\{x_n\}$ be a $t^{\omega}-$ Cauchy sequence in E_1, $\{f(\{x_n\})\}$ be a $t^{\omega}-$ Cauchy sequence in E_2 , the condition hold then there is $f(\{x_n\})$ in E_2. There exists $y = f(x) \in f(B) = A \in 2^{E_2}$ such that $t_2^{\omega}\big(f(\{x_n\}), f(B)\big) = 0$, that is $k t_1^{\omega}(\{x_n\}, B) < 0$,
for some $k \in [0,1]$, this impossible ■

Proposition 3.16: t^{ω}- normed app- space $(E, \|.\|, t^{\omega}{}_{\|.\|})$ is $t^{\omega}-$complete if and only if t^{ω}- metric approach space $(E, d_{\|.\|})$ is complete.

Proof: Let E be t^{ω}- normed app- space and that t^{ω} is generated by the $\|.\|$.

Let$\{x_n\}_{n=1}^{\infty}$ Caushy sequence in $(E, d_{\|.\|})$, then we have $d_{\|.\|}(x_n, x_m) = 0$, for all $n, m \in Z^+$ this implies that $t^{\omega}{}_{\|.\|}(\{x_n\}, A) = sup_{x_n\in X} \inf_{x_m\in A} d_{\|.\|}(x_n, x_m) = 0$, that is $\inf_{x_m\in A} t^{\omega}{}_{\|.\|}(\{x_n\}, A) = 0$, Then $\{x_n\}_{n=1}^{\infty}$ is $t^{\omega}-$Cauchy in $(E, \|.\|, t^{\omega}{}_{\|.\|})$ by proposition(3).

Since E is $t^{\omega}-$complete ,this implies, that there exists $x \in E$ for all $A \in 2^E$ such that $t^{\omega}{}_{\|.\|}(\{x_n\}, A) = 0$, for all $n \in Z^+$,
$d(x_n, x) = \inf_{x\in A} t^{\omega}(\{x_n\}, \{x\}) = 0$.

That is $\{x_n\} \to A$.

Conversely, suppose that $(E, d_{\|.\|})$ is complete and let $\{x_n\}$ is t^{ω} −Caushy sequence in t^{ω}-normed app-space. Then,

$0 = \inf_{x_m\in A} t^{\omega}{}_{\|.\|}(\{x_n\}, A) = sup_{x_n\in X} \inf_{x_m\in A} \|x_n - x_m\| = sup_{x_n\in X} \inf_{x_m\in A} d(x_n, x_m)$

$d_{\|.\|}(x_n, x_m) = \inf_{x_m\in A} t^{\omega}{}_{\|.\|}(\{x_n\}, \{x_m\}) = 0$.

$d_{\|.\|}(x_n, x_m) \to 0$ as $n \to \infty$. That is $\{x_n\}$ is Cauchy sequence in $\big(E, d_{\|.\|}\big)$.

$(X, d_{\|.\|})$ is complete, therefore $\{x_n\}$ is convergent sequence , there exists $x \in E$ such that $lim_{n\to\infty} x_n = x$. $d_{\|.\|}(x_n, x) = \inf_{x_m\in A} t^{\omega}{}_{\|.\|}(\{x_n\}, \{x\}) = 0$.

There exists $x \in A$, for all $A \in 2^E$,
such that $t^{\omega}{}_{\|.\|}(\{x_n\}, A) = sup_{x_n\in X} \inf_{x\in A} d_{\|.\|}(x_n, x) = 0$.

Hence, $(E, \|.\|, t^{\omega}{}_{\|.\|})$ is $t^{\omega}-$complete. ■

Proposition 3.17 : Let $(\chi, \|.\|, t^{\omega}{}_{\|.\|})$ be t^{ω}- normed vector app- space then the following are equivalent:

(1) $(\chi, \|.\|, t^{\omega}{}_{\|.\|})$ is t^{ω}- Banach app-space.

(2) (χ, t^{ω}) is complete.

Proof: That is clear by above corollary.

Remark 3.18 : Let $M = (\chi, d_{t^{\omega}})$ is t^{ω}-metric app-space, then M is a Hausdorff space.

4 CONCLUSION

In this paper, we proved that every t^{ω}-uniform normed app- space $(\chi, \|.\|, t^{\omega}{}_{\|.\|})$ is a Hausdorff space and we show that t^{ω}-normed app- space $(\chi, \|.\|, t^{\omega}{}_{\|.\|})$ is t^{ω}-complete if and only if a metric approach space $(E, d_{\|.\|})$ is complete. An example is given to explain that the metric app-space $(\chi, d_{t^{\omega}})$ is not normed app- space. We show that every finite –dimensional t^{ω}- normed app- space is complete. Also, the important and sufficient conditions are found to prove $(\chi, d_{t^{\omega}})$ is Banach app-space. Some other results that related to the convergent in t^{ω}-approach space and t^{ω}-normed approach space are proved.

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