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On the Commutative Path Hyperoperations

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Abstract

In this paper, we introduce a commutative path hyperoperation associated with a hypergraph. Also, we analyse some connections between commutative path hypergroupoids and hypergraphs.

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1 Introduction

Hypergraphs are systems of finite sets and form the most general concept in discrete mathematics. This branch of mathematics has developed very rapidly during the twentieth century. In [2], there is a very good presentation of graph and hypergraph theory.

Algebraic hyperstructures, in particular hypergroups, were introduced in 1934 by Marty, at the 8th Congress of Scandinavian Mathematicians and then it was developed by many researchers. Since then, hundreds of papers and several books have been written on this topic. Nowadays, there are many connections between hyperstructures and other branches of mathematics, leading to applications in hypergraphs, binary relations, combinatorics, artificial intelligence, automata and fuzzy sets. The concept of hypergroupoids deriving from binary relations, namely *C-hypergroupoids* were delineated by Corsini in [5].

A new class of hyperoperations, namely path hyperoperations that are obtained from binary relations and their connections with graph theory, were introduced by Kalampakas et al. [7, 9].

In the sequel, we introduce some preliminary results and definitions which will be needed in the subsequent section.

A *hypergraph* Γ is a pair (V, E), where $V = \{v_1, v_2, ..., v_n\}$ is a set of discrete elements known as vertices (or nodes) and $E = \{e_1, e_2, ..., e_m\}$ is a collection of arbitrary non-void subsets of V such that $\bigcup_j e_j = V$, known as edges (or hyperedges). A hypergraph is a generalization of an ordinary undirected graph, such that a hyperedge does not need to contain exactly two vertices, but can instead contain an arbitrary non-zero number of nodes. Also, an ordinary undirected graph (without self-loops) is a hypergraph such that every edge has exactly two vertices. Two vertices u and v are *adjacent* in $\Gamma = (V, E)$ if there is an edge $e \in E$ such that $u, v \in e$. If for two edges $e, f \in E$, $e \cap f \neq \emptyset$, then we say that e and f are *adjacent*.

[1]speaker

A vertex v and an edge e are *incident* if $v \in e$. We denote by $\Gamma(v)$ the *neighborhood* of a vertex v, i.e. $\Gamma(v) = \{u \in V : \{u, v\} \in E\}$. Given $v \in V$, denote the number of edges incident with v by $d(v)$; $d(v)$ is called the *degree* of v. A hypergraph in which all vertices have the same degree d is said to be *regular* of degree d or d*-regular*. The size, or the *cardinality*, $|e|$ of a hyperedge is the number of vertices in e. A hypergraph Γ is *simple* if there are no repeated edges and no edge properly contains another. A hypergraph is known as *uniform* or k*-uniform* if all the edges have cardinality k. Note that an ordinary graph with no isolated vertices is a 2-uniform hypergraph.

A *partial hypergraph* (*or subhypergraph*) $\Gamma' = (V', E')$ of a hypergraph $\Gamma = (V, E)$, denoted by $\Gamma' \subseteq \Gamma$, is a hypergraph such that $V' \subseteq V$ and $E' \subseteq E$. The partial hypergraph $\Gamma' = (V', E')$ is *induced* if $E' = \{e \in E \mid e \subseteq V'\}$. Induced hypergraphs will be denoted by $\langle V' \rangle$. A partial hypergraph of a simple hypergraph is always simple.

Let $\Gamma = (V, E)$ be a hypergraph. A *path* of length k in Γ is an alternating sequence $P_{v_1,v_{k+1}} = (v_1, e_1, v_2, ..., v_k, e_k, v_{k+1})$ in which $v_i \in V$ for each $i = 1, 2, ..., k+1$, $e_i \in E$, $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 1, 2, ..., k$ and $v_i \neq v_j$, $e_i \neq e_j$ for $i \neq j$. Also, a hypergraph is *connected* if there is a path between every pair of vertices. A *connected component* of a hypergraph is every maximal set of vertices such that are pairwise connected by a path. A *cycle* of length k is a sequence $(v_1, e_1, v_2, ..., v_k, e_k, v_1)$, such that P_{v_1,v_k} is a path. Also, a hypergraph is called *acyclic* if it does not contain any cycles.

Let H be a non-void set and $P^*(H)$ be the set of all non-void subsets of H. A *hyperoperation* on H is a map $* : H^2 \to P^*(H)$ and the couple $(H, *)$ is called a *partial hypergroupoid*. The structure $(H, *)$ is called a *non-partial hypergroupoid* if for every $x, y \in H$ we have $x * y \neq \emptyset$.

Let $\Gamma = (V, E)$ be a hypergraph. We define the *path hyperoperation* $\circ_\Gamma : V \times V \to P^*(V)$ for all $x, y \in V$ as follows:

$$x \circ_\Gamma y := \{z \in V \mid z \textit{ belongs to a path from } x \textit{ to } y\}.$$

The (partial) hypergroupoid $(V, \circ_\Gamma)$ is called the (partial) *path hypergroupoid* corresponding with Γ.

The hyperoperation "$\circ_\Gamma$" on V is called a *non-partial hyperoperation* if for all $x, y \in V$ we have $x \circ_\Gamma y \neq \emptyset$. In this case, the path hypergroupoid associated with Γ is called non-partial.

It is easy to check that, for any hypergraph $\Gamma = (V, E)$ and for all $x, y \in V$, the Corsini product $x \bullet_E y$ is a subset of $x \circ_\Gamma y$ i.e., $x \bullet_E y \subseteq x \circ_\Gamma y$.

In this paper, we introduce a commutative path hyperoperation associated with a hypergraph, also, we analyse some connections between commutative path hypergroupoids and hypergraphs.

2 Main results

By the definition of the path in a hypergraph we obtain the following results.

Proposition 2.1. *Let $\Gamma = (V, E)$ be a hypergraph. Then for any $x, y \in V$, $x \circ_\Gamma y \neq \emptyset$ if and only if there exists a path from x to y.*

Proposition 2.2. *([8]) Let $G = (V, E)$ be a graph, then the Corsini hyperoperation "$\bullet_E$" associated with G, is non-partial if and only if there exists a path with length 2 between any pair of vertices of G.*

This result allows us to prove the following corollary.

Corollary 2.3. *Let $\Gamma = (V, E)$ be a hypergraph and let "$\circ_\Gamma$" be the associated path hyperoperation with Γ. Then "$\circ_\Gamma$" is a non-partial hyperoperation if and only if for any $x, y \in V$, there exists a path from x to y.*

Proof. The path hyperoperation associated with Γ is a partial hyperoperation if and only if $x \circ_\Gamma y \neq \emptyset$ for all $x, y \in V$. Therefore, by Proposition 2.1, $x \circ_\Gamma y \neq \emptyset$ if and only if there exists a path from x to y. □

In the following proposition, we state the relation between a path hyperoperation and cycles of a hypergraph.

Theorem 2.4. *Let $\Gamma = (V, E)$ be a hypergraph and $x, y \in V$. Then $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$ if and only if x and y belong to at least one same cycle in Γ.*

Proof. Suppose that $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$, then by Proposition 2.1, there exists a path from x to y and a path from y to x, for any $x, y \in V$. Therefore, there exists at least one cycle passing through x and y.

Conversely, suppose that x and y belong to one common cycle in Γ. Clearly, this cycle can be separated into two paths one from x to y and other from y to x. Thus, by Proposition 2.1, $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$. □

Now, we obtain the following result by Theorem 2.4 .

Corollary 2.5. *Let $\Gamma = (V, E)$ be a hypergraph and $x, y \in V$. Then Γ is acyclic if and only if either the path hyperoperation is non-commutative or $x \circ_\Gamma y = y \circ_\Gamma x = \emptyset$ holds.*

This result allows us to prove the following theorem.

Theorem 2.6. *Let $\Gamma = (V, E)$ be a hypergraph such that for all $x, y \in V$, $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$. Then the associated path hypergroupoid "$\circ_\Gamma$" is commutative.*

Proof. We must prove that $x \circ_\Gamma y = y \circ_\Gamma x$. To prove this, we first need to show that $x \circ_\Gamma y \subseteq y \circ_\Gamma x$. Since $x \circ_\Gamma y \neq \emptyset$, it follows that there exists a vertex $z \in x \circ_\Gamma y$. Thus, there exists a path from x to y in Γ as follows:

$$x, e_0, x_1, e_1, ..., z, e_z, ..., x_n, e_n, y.$$

Also, since $y \circ_\Gamma x \neq \emptyset$, it follows that there exists a path from y to x in Γ as follows:

$$y, e_0^{'}, y_1, e_1^{'}, ..., y_n, e_n^{'}, x.$$

Thus, there exists a path from y to x, passing from z as follows:

$$y, e_0^{'}, y_1, ..., y_n, e_n^{'}, x, e_0, x_1, ..., z, e_z, ..., x_n, e_n, y, e_0^{'}, y_1, ..., y_n, e_n^{'}, x.$$

Therefore, $z \in y \circ_\Gamma x$ holds. Similarly, we can prove that $y \circ_\Gamma x \subseteq x \circ_\Gamma y$. This completes the proof. □

From Theorem 2.6, it is easy to show the following corollary.

Corollary 2.7. *Let $\Gamma = (V, E)$ be a hypergraph. Then any non-partial path hypergroupoid associated with Γ is commutative.*

Theorem 2.8. *Let $\Gamma = (V, E)$ be a hypergraph such that for all $x, y, z \in V$, $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma z \neq \emptyset$. If the associated path hyperoperation, "$\circ_\Gamma$" is commutative, then $x \circ_\Gamma y = y \circ_\Gamma z$.*

Proof. We first need to show that $x \circ_\Gamma y \subseteq y \circ_\Gamma z$. Since $x \circ_\Gamma y \neq \emptyset$, it follows that there is a vertex $w \in x \circ_\Gamma y$. Thus, there exists a path from x to y in Γ which contains w as follows:

$$x, e_0, x_1, e_1, ..., w, e_w, ..., x_n, e_n, y.$$

By hypothesis, since "$\circ_\Gamma$" is commutative, so $y \circ_\Gamma x \neq \emptyset$, thus there exists at least one path from y to x as follows:

$$y, e_0^{'}, y_1, e_1^{'}, ..., y_n, e_n^{'}, x.$$

Also, since $y \circ_\Gamma z \neq \emptyset$, thus there exists a path from y to z as follows:

$$y, e_0^{''}, z_1, e_1^{''}, ..., z_n, e_n^{''}, z.$$

Thus, there exists a path from y to z, passing from w as follows:

$$y, e_0^{'}, y_1, ..., y_n, e_n^{'}, x, e_0, x_1, ..., w, e_w, ..., x_n, e_n, y, e_0^{''}, z_1, ..., z_n, e_n^{''}, z.$$

Therefore, $w \in y \circ_\Gamma z$ and thus $x \circ_\Gamma y \subseteq y \circ_\Gamma z$. Similarly, we can prove that $y \circ_\Gamma z \subseteq x \circ_\Gamma y$. This completes the proof. □

For a given hypergraph $\Gamma = (V, E)$, we define a relation $\sim_V$ on the associated path hypergroupoid $(V, \circ_\Gamma)$ as follows: for all $x, y \in V$,

$$x \sim_V y \Longleftrightarrow x \circ_\Gamma y \neq \emptyset.$$

We have the following proposition:

Proposition 2.9. *Let $\Gamma = (V, E)$ be a hypergraph and "$\circ_\Gamma$" be the associated path hyperoperation with Γ. If "$\circ_\Gamma$" is commutative, then $\sim_V$ is an equivalence relation on V.*

Proof. By Theorem 2.6 clearly, $\sim_V$ satisfies the symmetric relation. Also, it is easy to see that the transitive relation holds by Theorem 2.8. To see that $\sim_V$ is reflexive, without loss of generality, we may suppose that Γ be a hypergraph without isolated vertices. It follows that for any vertex in V, there is at least one incoming hyperedge to it or one outgoing hyperedge from it. Thus, for any $x \in V$, there is at least one $y \in V$ such that $x \circ_\Gamma y \neq \emptyset$ or $y \circ_\Gamma x \neq \emptyset$. By hypothesis, since "$\circ_\Gamma$" is commutative, we have: $x \circ_\Gamma y = y \circ_\Gamma x \neq \emptyset$. It follows that there exists at least one path from x to y and at least one path from y to x. Therefore, there exists a path from x to x and thus we have $x \circ_\Gamma x \neq \emptyset$. Hence, $\sim_V$ is reflexive and thus $\sim_V$ is an equivalence relation on V. □

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