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On Semi-Direct Product of Groups

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Abstract

This paper presents a generalization of a theorem of the $C_m \rtimes C_4$ with $m \in \mathbb{Z}$ in some special cases to explain the structure of groups which isomorphic to $C_m \rtimes C_4$ supported Evidence-based and solved examples.

Keywords: Scott topology, Equivalence theory, Perfect map (between 3 to 5 keywords)

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1 Introduction

The direct product of groups is define for any G and H are groups. The direct product $G \times H$ of G and H is the set of all ordered pairs $\{(g, h) \mid g \in G, h \in H\}$ with the operation $(g_1, h_1)(g_2, h_2 = (g_1g_2, h_1h_2)$. In the definition, I have assumed that G and H are using multiplication notation. In general, the notation you use in $G \times H$ depends on the notation in the factors. In direct product of groups must satisfy the following condition, $G \trianglelefteq G \times H$, $H \trianglelefteq G \times H$ and $G \cap H = \{e\}$. So the semi-direct product is define by,for any two subgroups G and H of group $\mathbb{G}$ condition the one of those subgroups are normal subgroup of $\mathbb{G}$ we can say that $\mathbb{G}$ of semi-direct groups G and H where $G \trianglelefteq \mathbb{G}$.

Let $\mathbb{G}$ have subgroups G and H where $G \trianglelefteq \mathbb{G}$ and $GH = \mathbb{G}$ and $G \cap H = \{1\}$.We say that $\mathbb{G}$ is a semidirect product of G and H with G as the normal subgroup. This is written as $\mathbb{G} \cong G \rtimes H$

The Group Automorphism is an isomorphism from a group to itself. Let G be a group. A map φ from G to itself is termed an automorphism of G if it satisfies all of the following conditions:

1. φ is bijective
2. $\varphi(gh) = \varphi(g)\ \varphi(h)$ whenever g and h are both in G
3. $\varphi(e) = e$
4. $\varphi(g^{-1}) = (\varphi(g))^{-1}$

[1]speaker

The automorphisms group of cyclic group is abelian is define $Aut(G) = \times_{p\in\pi(n)} C_p$ where $p \in \pi(n)$. For more information we refer the following [1, 2, 3, 4, 5, 6, 7, 8]

Example 1.1. The multiplicative table of the group of 4th roots of unity $G = \{1,-1,i,-i\}$ can be written as above, which means that the map defined by

1 → 1, -1 → -1, i → -i, -i → i

	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	i	-i
-i	-i	i	-1	1
i	i	-i	1	-1

Example 1.2. Consider the cyclic group C_3, which can be presented as its Cayley table:

$(Z_3, +_3)$	$[0]_3$	$[1]_3$	$[2]_3$
$[0]_3$	$[0]_3$	$[1]_3$	$[2]_3$
$[1]_3$	$[1]_3$	$[2]_3$	$[0]_3$
$[2]_3$	$[2]_3$	$[0]_3$	$[1]_3$

The automorphism group of C_3 is given by: Aut$(C_3) = \{\phi, \theta\}$ where ϕ and θ are defined as: $\phi([0]_3) = [0]_3$, $\phi([1]_3) = [1]_3$, $\phi([2]_3) = [2]_3$, $\theta([0]_3) = [0]_3$, $\theta([1]_3) = [2]_3$, $\theta([2]_3) = [1]_3$. The Cayley table of Aut(C_3 is then:

	ϕ	θ
ϕ	ϕ	θ
θ	θ	ϕ

Definition 1.3. Semidirect Product of Groups: Let G have subgroups H and K where $H \lhd G$ and $HK = G$ and $H \cap K = \{1\}$.We say that G is a semidirect product of H and K with H as the normal subgroup. This is written as $G \cong H \rtimes K$

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Definition 1.5. Semidirect Product of Groups: Let H and K be groups and let K act on H. Let φ : $K \to$ Aut(H)be the action of K on H. Let G be a group of all pairs $(h$,$k)$ with $h \in H$ and $k \in K$ with the group operation

$$(h_1,k_1)\ (h_2,k_2) = (h_1\ \varphi_{(k_1)}\ h_2,k_1\ k_2)$$

Denote G by $H \rtimes_\varphi K$

Example 1.6. The dihedral group D_{2n} is isomorphic to $Z_n \rtimes_\varphi Z_2$ where $\varphi(1)(k)$ = n-k

Example 1.7. The infinite dihedral group D_∞ is isomorphic to $Z_n \rtimes_\varphi Z_2$, where $\varphi(1)(k)$ = -k

Example 1.8. Suppose $H \cong Z/3$ and $K \cong Z/2$. There is more than one possibility for G:

1. If K is also normal in G then we already know $G \cong H \times K$, which is abelian and isomorphic to $Z/6$.

2. On the other hand K might not be normal in G, for example G might be the symmetric group S_3, with H = {(1),(123),(132)} and K = {(1),(12)}.

Example 1.9. Consider D_{2n}. Let $K \subseteq D_{2n}$ be the subgroup consisting of the two reflections and let $H \subseteq D_{2n}$ be the subgroup of rotations. Note that $K \cong C_2$ and $H \cong C_n$. We can see that $H \triangleleft D_{2n}$ because H has index 2. Also $H \cap K = \{1\}$ and $HK = D_{2n}$. The conjugation action φ:$K \to$ Aut(H)is $\varphi(1)(h) = h$ and $\varphi(s)(h) = h^{-1}$ where s is the non identity element of K. So, we can conclude that $D_{2n} \cong H \rtimes_{\varphi} K \cong C_n \rtimes_{\varphi} C_2$.

2 The Preliminaries

Definition 2.1. Direct Product of Groups: Let G and H be groups. The direct product G X H of G and H is the set of all ordered pairs $\{(g, h) \mid g \in G, h \in H\}$ with the operation

$$(g_1 , h_1).(g_2 , h_2) = (g_1 \ g_2 , h_1 \ h_2)$$

In the definition, I've assumed that G and H are using multiplication notation. In general, the notation you use in G X H depends on the notation in the factors. Examples:

- $g_1 . g_2 \in G$ and $h_1 . h_2 \in H$ then $(g_1,h_1)(g_2,h_2)= (g_1 \ g_2, h_1 \ h_2) \in G \times H$
- $g_1 + g_2 \in G$ and $h_1 + h_2 \in H$ then $(g_1,h_1)(g_2,h_2)= (g_1 + g_2, h_1 + h_2) \in G \times H$
- $g_1 . g_2 \in G$ and $h_1 + h_2 \in H$ then $(g_1,h_1)(g_2,h_2)= (g_1 \ g_2, h_1 + h_2) \in G \times H$

Definition 2.2. Group Automorphism:A group automorphism is an isomorphism from a group to itself. Let G be a group. A map φ from G to itself is termed an automorphism of G if it satisfies all of the following conditions:

1. φ is bijective
2. $\varphi(gh) = \varphi(g)\ \varphi(h)$ whenever g and h are both in G
3. $\varphi(e) = e$
4. $\varphi(g^{-1}) = (\varphi(g))^{-1}$

The third and fourth condition follow from the first two (refer equivalence of definitions of group homomorphism).

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The automorphism group of C_3 is given by: Aut$(C_3) = \{\phi, \theta\}$ where ϕ and θ are defined as: $\phi([0]_3) = [0]_3$, $\phi([1]_3) = [1]_3$, $\phi([2]_3) = [2]_3$, $\theta([0]_3) = [0]_3$, $\theta([1]_3) = [2]_3$, $\theta([2]_3) = [1]_3$. The Cayley table of Aut(C_3 is then:

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Definition 2.7. Semidirect Product of Groups: Let H and K be groups and let K act on H. Let φ : $K \to$ Aut(H)be the action of K on H. Let G be a group of all pairs (h,k) with $h \in H$ and $k \in K$ with the group operation

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Denote G by $H \rtimes_\varphi K$

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2.1 Automorphism

An automorphism of a group G is an isomorphism of G with itself. The set of such is labelled Aut$G$. For two groups H and K and an action function $\phi : K \to Aut(H)$ of K on H by automorphism, the corresponding semidirect product $H \rtimes_\phi K$ is defined as follows: as a set it is $H \times K = \{(h, k) : h \in H, k \in K\}$. The group law on $H \rtimes_\phi K$ is

$(h, k)(h', k') = (h \rtimes_k (h'), kk')$

	ϕ	θ
ϕ	ϕ	θ
θ	θ	ϕ

1. $S_n \cong A_n \rtimes C_2$
2. $D_n \cong C_2 \rtimes C_n$

3 Main Results

Theorem 3.1. *Suppose that p is a prime number, the semi-direct product of groups C_p and C_4 be isomorphic to $C_p \rtimes C_4$.*

Proof. before proof above theorem we must be find the relation between the elements belong to semidirect product of C_4 and C_p It is clear that the Cyclic groups are define by $C_p = \langle a|a^p = e\rangle$ and $C_4 = \left\langle b|b^4 = e\right\rangle$. In the first we must prove that the group C_p is normal subgroup of $C_p \rtimes C_4 =< a, b|a^p = b^4 = e, b^3a^ib = a^{p-i} >$

$$\begin{aligned}
b^3a^ib &= a^{p-i}\\
b^3a^ib^2 &= a^{p-i}b\\
b^3a^ib^3 &= a^{p-i}b^2\\
b^3a^ib^4 &= a^{p-i}b^3\\
b^3a^ie &= a^{p-i}b^3\\
b^3a^i &= a^pa^{-i}b^3\\
b^3a^i &= a^{-i}b^3
\end{aligned}$$

since, for any element $a^tb^k \in C_p \rtimes C_4$, we can see,

$$\begin{aligned}
a^tb^kC_p(a^tb^k)^{-1} &= a^tb^k\left\langle a^i\right\rangle(a^tb^k)^{-1}\\
&= \left\langle a^tb^ka^i(a^tb^k)^{-1}\right\rangle\\
&= \left\langle a^tb^ka^ib^{-k}a^{-t}\right\rangle\\
&= \left\langle a^ta^ib^kb^{-k}a^{-t}\right\rangle\\
&= \left\langle a^ta^ia^{-t}\right\rangle = \left\langle a^i\right\rangle.
\end{aligned}$$

And $C_p \cap C_4 = \{e\}$, thus $|C_p \rtimes C_4| = \frac{|C_p||C_4|}{|C_p\cap C_4|} = 4p$. □

Theorem 3.2.

Suppose that $n = 4$ and $m = 2.k, k \in \mathbb{Z}$ are positive integer number, the semi-direct product $C_m \rtimes C_4$ be isomorphic to the follow in:

1. If $k = 1$, then $C_4 \rtimes C_m \cong D_8$;

Proof. by use theorem which states In D_n, every subgroups of $< r >$ is a normal subgroup of D_n; these are the subgroups $< r^d >$ for $d|n$ and have index $2d$. The describes all proper normal subgroups of D_n when n is odd, and the only additional proper normal subgroup when n is even are $< r^2, s >$ and $< r^2, rs >$ with index 2. $[D_8 : C_4] = |D_8|/|C_4| = 8/4 = 2$ since the index equal to 2 then $C_4 \triangleleft D_8$ and C_m where $m = 2k$ with $k = 1$ hence $C_2 \leq D_8$, from the Remark above in ,we get $C_4 \rtimes C_m \cong D_8$ □

2. If $k = 2^r, r > 0$, then $C_4 \rtimes C_m \cong C_4 \rtimes C_{2^{r+1}}$;

Proof. $C_4 \rtimes C_m \cong C_4 \rtimes C_{2k} \cong C_4 \rtimes C_{2*2^r}$

Product of two cyclic groups is cyclic iff their orders are Co-prime numbers or relatively prime numbers which means are those numbers that have their HCF (Highest Common Factor) as 1. In other words, two numbers are co-prime if they no common factor other than 1. Then we get

$C_4 \rtimes C_{2*2^r} \cong C_4 \rtimes C_{2^{r+1}}$ □

3. If $k = p$ is an odd prime number, then $C_4 \rtimes C_m \cong C_p \rtimes D_8$;

Proof. $C_4 \rtimes C_m \cong C_4 \rtimes C_{2k} \cong C_4 \rtimes C_{2*p} \cong C_4 \rtimes (C_2 \times C_p) \cong (C_4 \rtimes C_2) \times C_p \cong D_8 \rtimes C_p \cong C_p \times D_8$ □

4. If $k = 2^r p, r > 0$ and p is an odd prime number, then $C_4 \rtimes C_m \cong C_p \times (C_4 \rtimes C_{2^r})$;

Proof. $C_4 \rtimes C_m \cong C_4 \rtimes C_{2k} \cong C_4 \rtimes C_{2*2^r p} \cong C_4 \rtimes C_{2^{r+1}p} \cong C_4 \rtimes (C_{2^{r+1}} \times C_p) \cong (C_4 \rtimes C_{2^{r+1}}) \times C_p \cong C_p \times (C_4 \rtimes C_{2^{r+1}})$

□

In the final we can define presention of groups $C_m \rtimes C_4$ by the following:

Theorem 3.3. *Suppose that C_m and C_4 are cyclic groups and $m \in \mathbb{Z}$, the presentation the semi-direct groups is given by the following.*

$$\langle a, b | a^m = b^4, b^3 ab = a^{p-1} \rangle$$

Proof. Take $K = C_m = \{a\rangle$, $H = C_4 = \{b\rangle$. Then $Aut(K) = \langle \delta \rangle$ where $\delta(a) = a^{-1}$, it is clear that $Aut(C_m) \cong C_{m-1}$ if and only if $m = p$ where p is prime number. Define $\varphi : H \longrightarrow Aut(K)$ mapping $\varphi(y) = \delta$.

□

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