7th International Conference on
Combinatorics, Cryptography, Computer Science and Computing
November. 16-17, 2022

Nearly Endo –T–ABSO Submodule and Related Concepts

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ABSTRACT

In this work, we discuss Nearly Endo prime and Nearly Endo T-ABSO submodule concepts, observations, and the traits and characteristics of Endo T-ABSO and Jacobson radical in relation to one another. The relationship between Endo T-ABSO and Nearly Endo T-ABSO submodule were discovered. We are now introducing submodule of several types. The ramifications of this study can be used to create new Nearly Endo T-ABSO submodule based notions.
Keywords:T–ABSO submodule;Endo T–ABSO submodule ;Nearly Endo prime submodule; Nearly Endo T–ABSO submodule
Keywords: T–ABSO submodule ;Endo T–ABSO submodule ;Nearly Endo prime submodule; Nearly Endo T–ABSO submodule.

1 INTRODUCTION

In this work, W is a unitary G-module and G has the identity and would be a commutative ring. In 1983, Chin-Pi Lu [4] conducted research on the idea of a prime submodule of modules. Endo prime submodule [11], was first presented by SH. O. Dakheel in 2010. In 2011 [4], A. Y. Yousefian and F. Soheilnia developed the idea for 2-Absorbing submodule [2]. Abdulrahman Abdullah expanded this idea to include the Endo 2-absorbing submodule in 2015 [1].This study consists of two parts. In the first section, we introduce numerous ideas and some of their core attributes. The Nearly Endo-T-ABSO submodule's relationships, traits, and core discoveries are examined in the second section.

2 PRELIMINARIES

This section is present the features of various fundamental ideas.

Definition 2.1 [6]
A module W is said to be simple if, W≠0 and the only submod of W are 0 and W.

Definition 2.2 [6]
A submod $P \leq W$ is referred to as **minimal** (respectively **maximal**) submod of W if $P \neq 0$, $\forall B \leq W$, $[B \subsetneq P \Longrightarrow B=(0)]$ respectively $P \subsetneqq WM$, $\forall B \leq W$, $[P \subset B \Longrightarrow B=W]$

Definition 2 .3 [6]
An $G-$ module G is referred to as **a cyclic** if $m \in W$ such that
$W = < m> = Gm = \{rm : r \in G\}$.

Definition 2 .4 [*6*]
If a module W has a finite generating set, it is said to be finitely generated., say X, that is $W = \langle X \rangle$.

Definition 2.5 [6]

A submod P of a module W is referred to as a direct summand of W, for short $P \leq^{\oplus} W$ if, there exists a submod K of W such that $P + K = W$ and $P \cap K = 0$.

Definition 2.6 [4]

A proper submod P of an G − module W, is referred to as a prime submodif $a \in G$, $m \in W$, with $am \in P$ implies that $m \in P$ or $a \in (P :_G W)$.

Definition 2 .7 [11]

Aproper submod P of an G-moduleW is referred to as an Endo Prime Submod if $f \in End(W)$, $f(m) \in N$, $m \in W$ implies that $m \in P$ or $f(W) \subseteq P$.

Definition 2.8 [1]

A proper submod P of an GR − modle W, is referred to as T − ABSO submod if whenever $a, b \in G$, $x \in W$,With $abx \in P$ implies that $ax \in P$ or $bx \in P$ or $ab \in P$ or $ab \in (P :_G W)$.

Definition 2.9 [1]

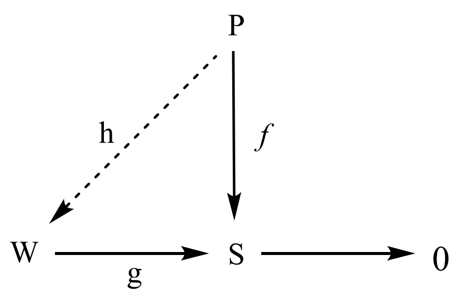
A proper submod P of an G-module W, is referred to as Endo T-ABSO submod if for each $f, g \in End(W)$, $m \in W$ With $(f \circ g)(m) \in P$ implies that $f(m) \in P$ or $g(m) \in P$ or $(f \circ g)(W) \subseteq P$.

Definition 2.10 [10]

Let W be an R-module. The Jacobson radical of W is denoted by J(W), and defined as the intersection of all maximal submod of W, or the sum of all small submod of W. If W has no maximal submod, then we set $J(W) = W$.

Definition 2.11 [9]

An G-module P is referred to as W-Projective modul if each pattern diagram :

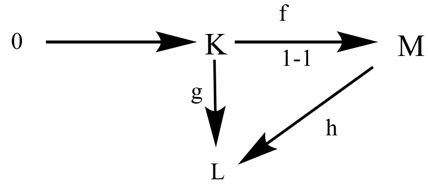


$$g \, o \, f = f$$

with exactrow can be extended commutatively via homomorphism $h: P \to W$ that is $goh = f$.

Definition 2.12 [9]

An G-module L is referred to as M-injective module if each pattern diagram :



With exact row can be exatended commutatively via a homomorphism $h: M \to L$ That is $hof = g$.

Definition 2.13 [10]

A submod P of a module W is said to be fully invariant if, $f(P) \subseteq P$ for all $f \in End_R(W)$.

Definition 2.14 [14]

A submod A of a module W is referred to as distributive if, $A \cap (B + C) = (A \cap B) + (A \cap C)$ or $(B \cap C) + A = (B + A) \cap (C + A)$ for all submod B, C of W. A module W is said to be distributive if, all submod of W are distributive.

Definition 2.15 [15]

Let G be a commutative ring with unity and W be a unital G-module. Then W is called **a multiplication module** if for each submod P of W, there exists an ideal I of G like that P=IG .

Corollary 2.16 [3]

Every finitely generated multiplication G-modul W is Scalar module.

Definition 2.17 [3]

An G- module W is referred to as a Scalar module if for each $f \in End(W)$, there exists $r \in G$ such that $f(m) = rm$, for $m \in W$

Definition 2.18 [12]

A ringR is **a domain**if for all a, b ∈ R, ab = 0, then a = 0 or b = 0, i.e., R has no zero divisor. A commutative domain with unity is called an **integral domain**.

Definition 2.19 [7]

A moduleWover an integral domain G is referred to as torsion module torsion free if $T(W) = W$. T(W)=0, respectively where $T(W) = \{m \in W: \text{there exists } r \in G, rm = 0, r \neq 0\}$

Definition 2.20 [13]

Let G be integral domain, an G-module W means divisible if, $rW = W$ forevery non-zero element $r \in R$.

3 *Nearly – Endo– T– ABSO Submod:*

Several theorems, relationships, comments, instances, and statements are offered, together with the Nearly Endo T-ABSO submod idea, further definitions, and the required to establish them.

Definition 3.1

A proper submod P of an G- module W is called Nearly EndoPrime submod (by briefly N-E-Prime submod of W, if when even $f \in End(W)$, $m \in W$ such that $f(m) \in P$, implies that $either$ $m \in P + J(w)$ or $f(w) \subseteq P + J(w)$.

Example 3.2

Let Z_6 *as* $Z - module$. $P = (\overline{2})$ *is* $N - E - Prime$ *submodule since if* $f(x) = 2x$, $\forall x \in Z_6$, $f \in Endo(Z_6)$ *and* $J(Z_6) = (\overline{2}) \cap (\overline{3}) = (\overline{0})$, $f(2) = 2(2) = 4 \in P = (\overline{2})$, *implies that either* $2 \in (\overline{2}) + J(Z_6) = (\overline{2})$ *or* $f(Z_6) \subseteq (\overline{2}) + J(Z_6) = (\overline{2})$

Definition 3.3

A proper submod P of an G-module W, is called Nearly EndoT-ABSO submod (by briefly N-E-T-ABSO submodof W, if whenever $f, g \in Endo(W), m \in W$ such that $(f\text{og})(m) \in P$, implies that either $f(m) \in P + J(W)$ or $g(m) \in P + J(W)$ or $(f\text{og})(W) \subseteq P + J(W)$.

Example 3.4

Consider Z_{15} as Z-module, P=$(\overline{3}) = \{0,3,6,9,12\}$ is N-E-T-ABSO submod, since if $f, g \in Endo(Z_{15})$, $m \in Z_{15}$, $f(x) = 3x$, $\forall x \in Z_{15}$ $g(x) = x$ $\forall x \in Z_{15}$ and $J(Z_{15}) = (\overline{3}) \cap (\overline{5}) = (\overline{0})$ $(f\text{og})(3) = f(3) = 9 \in P$, implies that either $f(3) = 3(3) = 9 \in P + J(Z_{15}) = (\overline{3})$ *or* $g(3) = 3 \in P + J(Z_{15}) = (\overline{3})$ or $(f\text{og})(Z_{15}) \subseteq P + J(Z_{15}) = (\overline{3})$. So that$(f \circ g)(Z_{15}) \subseteq P + J(Z_{15}) = (\overline{3})$

Remark and Examples 3.5

1) Every N-E- Prime Submod is N-E-T-ABSO submod , but the converse incorrectin general, for example: Consider Z_6 *as* $Z - module$, $(\overline{2})$ is a submod of Z_6, $(\overline{2})$ is N-E-T-ABSO submod , since if $f, g \in End(Z_6)$, $f(x) = x + 1$ and $g(x) = 3$, $\forall x \in Z_6$, $(f \circ g)(2) = f(g(2)) = 4 \in (\overline{2})$ implies that$(f \circ g)(Z_6) \subseteq (\overline{2})$ $J(Z_6)$ where $J(Z_6) = (\overline{2}) \cap \overline{(3)} = (\overline{0})$. So that $(f \circ g)(Z_6) \subseteq (\overline{2}) + J(Z_6)$. But it is not N-E-prime submod of Z_6, $f(3) = 4 \in (\overline{2})$, implies that either $3 \notin (\overline{2}) + J(Z_6) = (\overline{2})$ or $f(Z_6) \nsubseteq (\overline{2}) + J(Z_6) = (\overline{2})$.

2) Let P, S be two submods of an G-module W and $P < S$. If S is N- E-T-ABSO submod of W, then P is not necessary that N-E-T-ABSO submod of W, for example: consider Z_{24} as Z-module,Take $S = (\overline{4})$, $P = (\overline{12})$, $\forall f, g \in Endo(Z_{24}), f(x) = x - 2$, $g(x) = 2x$, $\forall x \in Z_{24}$ where $J(Z_{24}) = (\overline{2}) \cap (\overline{3}) = (\overline{6})$ $S = (\overline{4})$ is N-E-T-ABSO submod of $W = Z_{24}$ since $(f \circ g)(7) = f(g(7)) = 12 \in S$ implies that $g(7) = 14 \in S + J(Z_{24}) = (\overline{2})$, but $P = (\overline{12})$ is not N-E-T- ABSO submod of $W = Z_{24}$ since $(f \circ g)(7) = f(g(7)) = 12 \in P$, then $f(7) = 5 \notin P + J(Z_{24}) = (\overline{6})$, $g(7) = 14 \notin P + J(Z_{24}) = (\overline{6})$ and $(f \circ g)(Z_{24}) \nsubseteq P + J(Z_{24})$ Where $(f \circ g)(6) = 10 \notin P + J(Z_{24}) = (\overline{6})$

3) Every E-Prime submod of an G-module W is N- E-Prime submod.

Proof : Let P be E-Prime submod of and $f \in End(m)$, $m \in W$ such that $f(m) \in P$, to prove $m \in P + J(W)$ *or* $f(W) \subseteq P + J(W)$. Since P is E-Prime submod, then $m \in P$ or $f(W) \subseteq P$, hence $m \in P + J(W)$ $f(W) \subseteq P + J(W)$ since $P \subseteq P + J(W)$ Thus P is N-E-Prime submod of W

But theconverse of (3) is nottrue generally, for example: Consider Z_{20} as Z-module, $P = (\overline{4})$ is N-E-T-ABSO submod since if $f \in End(Z_{20}, f(x) = x - 2$, $\forall x \in Z_{20}$ where $J(Z_{20}) = (\overline{2}) \cap (\overline{5}) = (\overline{10})$, $f(6) = 4 \in P = (\overline{4})$, then either $6 \in P + J(Z_{20}) = (\overline{2})$ or $f(Z_{20}) \subseteq P$, but P is not E-Prime submod, since $f(6) = 4 \in P$, then $6 \notin P = (\overline{4})$ and $f(Z_{20}) \nsubseteq P = (\overline{4})$ where $f(3) = 1 \notin P$

4) Every E-T-ABSO submod of an G-module W is N-E-T-ABSO submod.

Proof: Let P be E-T-ABSO submod of an G-module W and $f, g \in End(W)$, $m \in W$ such that $(f \circ g)(m) \in P$, but P is E-T-ABSO submod of W, then $f(m) \in P$ or $g(m) \in P$ or $(f \circ g)(W) \subseteq P$, hence $f(m) \in P + J(W)$ or $g(m) \in P + J(W)$ or $(f \circ g)(W) \subseteq P + J(W)$ since P$\subseteq$ P+$J(W)$. Thus P is N-E-T-ABSO submod.

But he converse of (4) incorrect in general, forexample: consider Z_{24} as $Z -$ module, $P = (\overline{8})$ is N-E-T-ABSO submod, since if $f, g \in End(Z_{24}), f(x) = x - 2$, $g(x) = x + 1, \forall x \in Z_{24}$ where $J(Z_{24}) = (\overline{2}) \cap (\overline{3}) = (\overline{6})$ such that $(f \circ g)(9) = f(g(9))$ $= 8 \in P = (\overline{8})$, then $g(9) = 10 \in P + J(Z_{24}) = (\overline{2})$, but $f(9) = 7 \notin P$ or $g(9) = 10 \notin P$ or $(f \circ g)(Z_{24}) \nsubseteq P$ where

$(f \circ g)(10) = f(g(10)) = 9 \notin P$. So that P is N-E-T- ABSO submod of W.

5) Let P, S be two submods of an G-module W, and $P \subset S$. If P is N-E-T-ABSO submod of W, then P is N-E-T-ABSO submod of S with $J(W) \subseteq J(S)$

Proof: Let $(f \circ g)(m) \in P,\ \forall\, m \in S$ since $S < W, so\ m \in W, f \in End(W)$, Since P is N-E-TABSO submod of W, then either $f(m) \in P + J(W)$ or $g(m) \in P + J(W)$ or $(f \circ g)(W) \subseteq P + J(W)$, since $J(W) \subseteq J(S)$, hence $f(m) \in P + J(S)$ or $g(m) \in P + J(S)$ or $(f \circ g)(W) \subseteq P + J(S)$, but $S < W$, so that $(f \circ g)(S) \subseteq (f \circ g)(W)$, hence $(f \circ g)(S) \subseteq P + J(S)$. Thus P is N-E-T-ABSO submod of S.

6) The intersection of two N- E- T-ABSO submod not be N-E-T-ABSO submod, for example: consider Z_{12} as Z-module take $P = (\bar{4})$, $S = (\bar{3})$ are N-E-T-ABSO submods of Z_{12}, since $\forall\, f, g \in Endo(Z_{12})$, $f(x) = x - 3$, $g(x) = x - 2$, $\forall\, x \in Z_{12}$ where $J(Z_{12}) = (\bar{2}) \cap (\bar{3}) = (\bar{6})$ such that $(f \circ g)(5) = f(g(5)) = 0 \in (\bar{4})$, then $f(5) = 2 \in (\bar{4}) + J(Z_{24}) = (\bar{2})$, also $(f \circ g)(5) = f(g(5)) = 0 \in (\bar{3})$, then $g(5) = 3 \in (\bar{3}) + J(Z_{12}) = (\bar{3})$. But $(\bar{4}) \cap (\bar{3}) = (\bar{0})$ is not N-E-T-ABSO submods of Z_{12}, since $(f \circ g)(5) = f(g(5)) = 0 \in (\bar{0})$, then $f(5) = 2 \notin (\bar{0}) + J(Z_{12}) = (\bar{6})$, $g(5) = 2 \notin (\bar{0}) + J(Z_{12}) = (\bar{6})$ and $(f \circ g)(Z_{12}) \nsubseteq (\bar{0}) + J(Z_{12}) = (\bar{6})$, Where $(f \circ g)(6) = 1 \notin (\bar{0}) + J(Z_{12}) = (\bar{6})$

Proposition 3.6

Let P be N-E-T-ABSO submod of an G-module W is Scalar module and $J(W) \subseteq P$ if and only if P is T-ABSO submod of W**.**

Proof: $(\Longrightarrow)$ Let $f, g \in \mathrm{Endo}(W)$, $\forall m \in W$ such that $f(m) = am$, $g(m) = bm$ $(f \circ g)(M) = abm \subseteq P$, *since* W is Scalar modul but P is N-E-T-ABSO submod of W, then either $am = f(m) \in P + J(W)$ *or* $bm = g(m) \in P + J(W)$ *or* $(f \circ g)(W) = abW \subseteq P + J(W)$, hence $f(m) = am \in P$ or $bm = g(m) \in P$ or $abW = (f \circ g)(W) \subseteq P$, so that $ab \in (P :_G W)$

Since $J(W) \subseteq P$. Then P is T-ABSO submod of W.

$(\Longleftarrow)$Let $(f \circ g)(m) \in P$ *where* $f, g \in \mathrm{Endo}(W)$, $\forall m \in W$, *since* W is Scalar module , then there exist a,b $\in R$ like that $am = f(m)$, $bm = g(m)$ for each $m \in W$, but P is T-ABSO submod of W, then $(f \circ g)(m) = abm \subseteq P$ implies that either $am = f(m) \in P$ *or* $bm = g(m) \in P$ or $abW \subseteq P$, Since $J(W) \subseteq P$ hence $f(m) \in P + J(W)$ or $g(m) \in P + J(W)$ or $(f \circ g)(W) \subseteq P + J(W)$.Then P is N-E- T-ABSO submod of W.

Remark 3.7

If delete the condition of Scalar module the converse of Proposition 3.6 is not true in general, forexample: Consider Z_{12} as Z-module, $P = (\bar{6})$ is T- ABSO submod, $2 \cdot 6 \cdot (\bar{1}) = 12 = 0 \in P = (\bar{6})$, implies that $6 \cdot (\bar{1}) \in P$ or $2 \cdot 6 = 12 = 0 \in (P :_Z Z_{12}) = 6Z$. So that P is T-ABSO submod of Z_{12}. But P is not N-E-T-ABSO submod of Z_{12} since if $f, g \in Endo(Z_{12})$, $f(x) = x - 3$, $g(x) = 3x$ such that $(f \circ g)(5) = f(g(5)) = 12 = 0 \in P$, where $J(Z_{12}) = (\bar{2}) \cap (\bar{3}) = (\bar{6})$, then $f(5) = 2 \notin P + J(Z_{12}) = (\bar{6})$ *and* $g(5) = 15 = 3 \notin P + J(Z_{12}) = (\bar{6})$ and $(f \circ g)(Z_{12}) \nsubseteq P + J(Z_{12})$ Where $(f \circ g)(4) = f(g(4)) = 9 \notin (\bar{6}) = P + J(Z_{12})$

Corollary 3.8

Let P a proper submod of a finitely generated multiplication G-module W $J(W) \subseteq P$. then P is T-ABSO submod if and only if P is N-E-T-ABSO submod

Proof: By Proposition3.6 and Corollary 2.16 we get the result .

Proposition 3.9

Let W be a nonzero modle over an integral domainG. If W torsion module then W has no N-E-T-ABSO submod

Proof : Suppose that W has a T-ABSO submod say P since W istorsion module, then $\left(\frac{W}{p}\right)$ is torsionG-module, also we show that $\left(\frac{W}{p}\right)$ is torsion free G-module as follows:

let $0 \neq (m+P) \in \left(\frac{G}{p}\right)$, suppose $r(m+P) = P$ for some $r \in G$

if r≠ 0, then $rm \in P$ and $m \notin P$ since M is divisible, then $\frac{W}{r} = W$ so $m = rm_1$, for some $m \in W$ there fore $rm = r^2 m_1 \in P,\ but\ P\ is\ T-ABSO, then\ rm_1 \in P\ or\ r^2 \in (P{:}W)$.

If $rm_1 \in P,\ then\ m \in P\ which\ is\ a$ Contradiction.

If $r^2 \in (P{:}W)$ then $r^2 w \subseteq P$, but w is divisible, so $r^2 W = W$ there fore W = P whichis Contradiction.

If $r = 0$ and $\left(\frac{W}{P}\right)$ is torsionand torsion freewhich imply that $\left(\frac{W}{P}\right) = 0$, so $W = P$ which is Contradiction. Hence w has no T-ABSO submod so W has no N-E-T-ABSO submod.

Example 3.10

Z_{p^∞} as Z-module satisfies the requirement of Proposition3.9 hence Z_{p^∞} has no N-E-T-ABSO $submodule$. Also we can show this as follows:

first we show that (0) in not T-ABSO submod of Z_{p^∞} .

let $pp\left(\frac{1}{p2} + Z\right) = Z = 0$ but $p\left(\frac{1}{p2} + Z\right) \neq Z \neq 0$ and

$P^2 \notin \left(0{:}Z_{p^\infty}\right) = 0$. Now, for $0 \neq K < Z_{p^\infty}$, let $K = \left(\frac{1}{pi} + Z\right)$ for some $i \in Z_+$ so $p.p(\frac{1}{pi+2} + Z) \subseteq K$. but $p(\frac{1}{pi+2} + Z) \nsubseteq K$. and $p^z \notin (K{:}Z_{p^\infty}) = 0$. Thus K is not T-ABSO submod of Z_{p^∞}. hence every proper submod of Z_{p^∞} is not N-E-T-ABSO submod. That is Z_{p^∞} has no N-E-T-ABSO submod.

Proposition 3.11

Let $f{:}W \to W$ be an G-homomorphism. If p is fully invariant E-T- ABSO submod of W, such that $f(W) \nsubseteq P$. Then $f^{-1}(p)$ is N-E-T-ABSO submod of W.

Proof : Since p is a proper subm0d of W, So $f^{-1}(p)$ is submod of W. Let $f, g \in \mathrm{Endo}(W)$, $\forall m \in W$ such that $(h \circ g)(m) \in f^{-1}(p), then\ f(h \circ g)(m) \in p\quad so\ (f \circ h)g(m) \in p.\ since$ P is E-T-ABSO submod of W then either $f(g(m)) \in P\ or\ h(g(m)) \in P\ or\ f(h(W)) \subseteq P$.

Case 1: If $f(g(m)) \in P, f^{-1}\left[f(g(m)) \in P\right]$ then $g(m) \in f^{-1}(p)$ $g(m) \in f^{-1}(p) \subseteq f^{-1}(p) + f^{-1}(J(W))$

Case 2: If $h(g(m)) \in P$ but p is E-T-ABSO submod of W then $h(m) \in p$ or $g(m) \in p$ or $h(g(m)) \subseteq P$. If $h(m) \in p$ then $f(h(m)) \in f(p) \subseteq P$ since p is fully invariant $f(h(m)) \in p$, $f^{-1}[f(h(m)) \in p]$ so $h(m) \in f^{-1}(p) \subseteq f^{-1}(p) + f^{-1}(J(W))\ then\ h(m) \in f^{-1}(p) + f^{-1}(J(W))$. If $g(m) \in p$ then $f(g(m)) \in f(p) \subseteq P$, since p is fully invariant $f(g(m)) \in p$, so $g(m) \in f^{-1}(p) \subseteq f^{-1}(p) + f^{-1}(J(W))\ then\ g(m) \in f^{-1}(p) + f^{-1}(J(W))$. If $h(g(m)) \subseteq p$ then $f(h(g(m)) \in f(p) \subseteq P$ since p is fully invariant $f(h(g(m)) \subseteq P$, $f^{-1}[f(h(g(m)) \in p]$ so $h(g(m)) \in f^{-1}(p) \subseteq f^{-1}(p) + f^{-1}(J(W))\ then\ h(g(m)) \in f^{-1}(p) + f^{-1}(J(W))$.

Case3: If $f(h(W) \subseteq P$, $f^{-1}[f(h(W) \subseteq P]$ then $h(W) \subseteq f^{-1}(p) \subseteq f^{-1}(p) + f^{-1}(J(W))\ so\ h(W) \in f^{-1}(p)$. Thus $f^{-1}(p)$ is an N-E-T-ABSO submod of W

Proposition 3.12

Let P be a proper submod of G-module of W, Let S be afully invariant submod of G-module W and contained in P. If $\frac{P}{S}$ is an E-T- ABSO submod of $\frac{W}{S}$. then P is an N-E-T-ABSO submod of W.

Proof : Let $f, h \in EndoW$, $m \in W$ such that $foh(m) \in P$. Define f_1, h_1: $\frac{W}{S} \to \frac{W}{S}$ by $f_1(m + S) = f(m) + S$, $h_1(m + S) = h(m) + S$ $for\ each\ m \in W$. It is clear that f_1, h_1 are well-defined. Now $f \circ h(m + S) = f_1(h_1(m + S)) = f_1(m) + S$, $f_1(h(m) + S) = f \circ h(m) + S \in \frac{P}{S}$

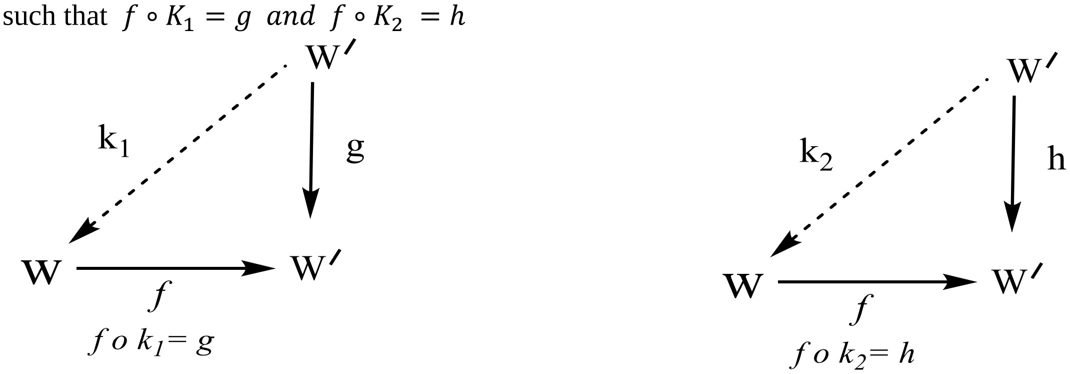
As $\frac{P}{S}$ is E-T-ABSO submod of $\frac{W}{S}$. Then either $h_1(m + S) \in \frac{P}{S}$ or $f_1(m) + S$ or $f_1(h_1(\frac{W}{S})) \subseteq \frac{P}{S}$, $h(m) + S \in \frac{P}{S} + J(\frac{W}{S})$ or $f(m) + S \in \frac{P}{S} + J(\frac{W}{S})$ or $f(h(W) + s) \subseteq \frac{P}{S} + J(\frac{M}{K})$. therefore $h(m) \in P + J(W)$ or $f(m) \in P + J(W)$ or $f(h(W)) \subseteq P + J(W)$. thus N is an N-E-T-ABSO submod of W.

Theorem 3.13

Let P be a fully invariant N-E-T-ABSO submod of G-module W,

Let $f: W \to W'$ be epimorphism such that $Ker\ f \subseteq P$. Then $f(P)$ is N-E-T- ABSO submod of W' where W ' is W '-projective Module.

Proof: Let $g, h \in EndW'$, $m' \in W'$ such that $(g \circ h)(m') \in f(P)$ since f is epimorphism then $m' = f(m)$ for some $m \in P$ since W' is W '-projective Module , there exist $K_1, K_2: W' \to W$, such that $f \circ K_1 = g$ and $f \circ K_2 = h$



Now consider the following digrams

$(g \circ h)(m') = (f \circ K_2) \circ (m') \in f(P) = f[(f \circ K_1 \circ K_2)(m')] \in f(P)$

Then$(K_1 \circ f \circ K_2)(m') \in P + Kerf$,since$Kerf \subseteq P, and\ m' = f(m)$, so $(K_1 \circ f \circ K_2)(f(m)) \in P$. That is $(K_1 \circ f)(K_2 \circ f)(m)) \in P$, since P is Endo-T-ABSO submod of W , then either $(K_1 \circ f)(m) \in P + J(W)$ ***or*** $(K_2 \circ f)(m) \in P + J(W)$ ***or*** $(K_1 \circ f) \circ (K_2 \circ f)(W) \subseteq P + J(W)$. If $(K_1 \circ f)(m) \in P + J(W)$, then $(K_1(f(m)) \in P + J(W), i.e$
$K_1(m') \in P + J(W)$, so $f(K_1(m')) \in f(P) + f(J(W))$ thus $g(m') \in f(P) + f(J(W))$.

If $(K_2 \circ f)(m) \in P + J(W), then (K_2(f(m)) \in P + J(W), i.e\ [K_2(m') \in P + J(W)], so\ f(K_2(m')) \in f(P) + f(J(W))\ thus\ h(m') \in f(P) + f(J(W))$

If $(K_1 \circ f) \circ (K_2 \circ f)(W) \subseteq P + J(W)$, then $(K_1 \circ f \circ K_2) f(W)) \subseteq P + J(W)$, Since $f(W) = W'$, $(K_1 \circ f \circ K_2)(W') \subseteq P + J(W)$, $since\ (foK_2) = h$, $then$ $(K_1 \circ h)(W') \subseteq P + J(W)$. So $f[(K_1 \circ h)(W')] \subseteq f(P) + f(J(W))$, $i.e$ $((f \circ K_1) \circ h)(W') \subseteq f(P) + f(J(W))$. Since $f \circ K_1 = g$, then $(goh)(W') \subseteq f(P) + f(J(W))$.

Thus $f(N)\ is\ an\ N - E - T - ABSO\ submodule\ of\ W'$.

Corollary 3.14

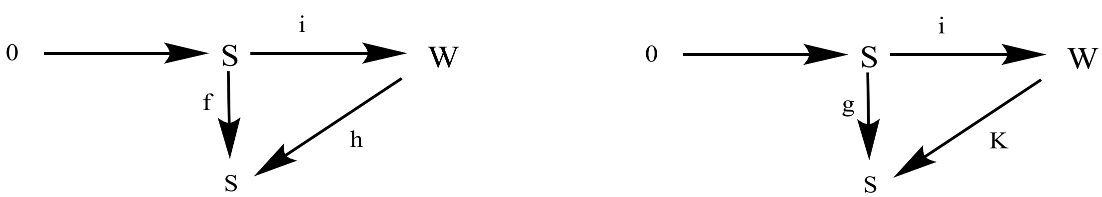
Let P be a fully invariant N-E-T-ABSO submod of G-module W. If K is submod of W, such that $K \subseteq P$, *then* $\frac{P}{K}$ is N-E-T- ABSO submod of $\frac{W}{K}$, Provided $\frac{W}{K}$, is an W-projective Module.

***Proof* :** $\boldsymbol{\pi : W \rightarrow \frac{W}{K}}$, $\boldsymbol{\pi(P) = \frac{P}{K}}$, then by Theorem 3.17, we get result.

Theorem 3.15

Let P be an N-E-T-ABSO submod of an G-module W and S is a submod of W , which is W-injective submod such that $J(W) \subseteq J(S)$ *and* $J(S)$ is distributive submod. Then either $S \subseteq P$ *or* $S \cap P$ is N-E-TABSO submod of S.

***Proof* :** Suppose that $S \nsubseteq P$, *then* $S \cap P \subsetneq S$, *let* $f, g \in EndS$, $x \in S$ *such that* $fog(x) \in S \cap P$ *suppose* $g(x) \notin S \cap P + J(S)$, *then* $g(x) \notin [S + J(S)] \cap [P + J(S)]$, *since* $J(S)$ *is* distributive submod , hence $g(x) \notin P + J(S)$, so that $g(x) \notin P + J(W)$, since $J(W) \subseteq J(S)$, to prove $f(x) \in S \cap P + J(S)$ *or* $f \circ g(S) \subseteq S \cap P + J(S)$, since S is W-injective submod, then there exists h , $K: W \rightarrow S$ as in the figure :



Now consider the following diagrams

Where I is the inclusion mapping and $hoi = f$, $koi = g$ clearly that h, k $\in EndS$, But $f \circ g(x) = (f \circ i) \circ (k \circ i)](x) = [(h \circ i \circ k)](x) = (h \circ i)[k(x)] = (h \circ k)(x) \in P$, since p is N-E-TABSO submod of W and $g(x) \notin P + J(W)$ implies that $k(x) \notin P + J(W)$. Then either $h(x) \in P + J(W)$ or $h \circ k(W) \in P + J(W)$.

If $h(x) \in P + J(W)$, then $h(x) \in S \cap P + J(W)$ since $h(x) \in S$, so $h(x) \in S + J(W)$ but $J(W) \subseteq J(S)$, *hence* $h(x) \in S \cap P + J(S)$. Thus $f(x) \in S \cap P + J(S)$.

Now, if $(h \circ k)(W) \subseteq P + J(W)$.

As $f \circ g(S) = (h \circ i) \circ (k \circ i)](S) = [(h \circ i \circ k)](S) = (h \circ i)[k(S)] = (h \circ k)(S) \subseteq P + J(W)$, *then* $f \circ g(S) \subseteq P + J(S)$, *since* $J(W) \subseteq J(S)$. Also $(f \circ g)(S) \subseteq S$, hence $fog(S) \subseteq S + J(S)$. Thus $(f \circ g)(S) \subseteq \left(P + J(S)\right) \cap \left(S + J(S)\right)$, so that $(f \circ g)(S) \subseteq (P \cap S) + J(S)$, since $J(S)$ is distributive submod . Therefore $S \cap P$ is N-E-TABSO submod of S.

CONCLUSIONS

As a new generalization of the T-ABSO submod, the N-E-T-ABSO submod are introduced. The following are the study's main findings:

1) Every N-E- Prime Submod is N-E-T-ABSO submod, but the converse incorrcet in general, see Remarks and Examples 2.1.3
2) Let P, S be two submods of an G-module W and P< S. If S is N -E-T-ABSO submod of W, then P is not necessary that N-E-T-ABSO submod of W.
3) Every E-Prime submod of an G-module W is N- E-Prime submod .
4) Every E-T-ABSO submod of an G-module W is N-E-T-ABSO submod. But the converse incorrect in general..
5) Let P, S be two submods of an G-module W, and P<S. If P is N-E- T-ABSO submod of W . then P is N-E-T-ABSO submod of S with $J(W) \subseteq J(S)$.
6) The intersection of two N- E- T-ABSO submod not be N-E-T-ABSO submod.
7) Let P be N-E-T-ABSO submod of an G-module W is Scalar module and $J(W) \subseteq P$ if and only if P is T-ABSO submod of W.
8) Let W be a $nonzero\ module$ over an $integral\ domain$. If W torsion module then W has no $N-Endo-T-ABSO\ submodule$.
9) Let$f$: W → W be an G-homomorphism. If p is $fully\ invariant$ Endo-T- ABSO submod of W, $such\ that\ f(W) \nsubseteq P$. $Then\ f^{-1}(p)$ is also an N-E-T-ABSO submod of W.
10) Let P be a fully invariant N-E-T-ABSO submod of G-module W,
Let be $f: W \to W'$ be epimorphism such that $Ker\ f \subseteq P$. Then f(P) is N-E-T- ABSO submod of W' where W ' is W '-projective Module..

11) Let P be an N-E-T-ABSO $\boldsymbol{submod}$ *of an G-m*$\boldsymbol{odule}$ *W* $\boldsymbol{and}$ *S is a* $\boldsymbol{submod}$ *of W , which is W-*$\boldsymbol{injective}$ *sub*$\boldsymbol{mod}$ *such that* $\boldsymbol{J(W) \subseteq J(S)}$ $\boldsymbol{and}$ $\boldsymbol{J(S)}$ *is distributive submod. Then either* $\boldsymbol{S \subseteq P}$ $\boldsymbol{or}$ $\boldsymbol{S \cap P}$ *is N-E-TABSO submod of S*

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