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STRONGLY EDGE DISTANCE-BALANCED GRAPH PRODUCTS

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Abstract

In a graph A for every arbitrary edge $f = \alpha\beta \in E(A)$ and every two integers i, j we consider $\acute{D}^i_j(f) = \{\acute{f} \in E(A) | d_A(\acute{f}, \alpha) = i, d_A(\acute{f}, \beta) = j\}$. In this article, we define A strongly edge distance-balanced ($SEDB$), whenever for each edge $f = \alpha\beta$ and each integer $i \geq 1$, $\acute{D}^i_{i-1}(f) = \acute{D}^{i-1}_i(f)$ and then verify some its properties. Moreover, we investigate cartesian and lexicographic products for such graphs.

Keywords: edge distance-balanced graphs, strongly edge distance-balanced graphs, graph products, cartesian and lexicographic products

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1 Introduction

The notion of graph is an essential tool to make use of the modeling of the phenomena and it is taken into consideration in many studies in a recent decades. One of the optimal uses of graphs theory is to classify graphs based on discriminating quality. This phenomenon can be best observed in distance-balanced graphs has been determined by [7]. Also, it is investigated in some papers, we refer the reader to ([2],[3],[4],[7]-[11]) and references therein. We consider A is a connected, finite and undirected graph throughout of this paper, in which its vertex set is $V(A)$ and its edge set is $E(A)$. In graph A, the distance among vertices $\alpha, \beta \in V(A)$ is introduced the number of edges in the least distance joining them and it is indicated by $d_A(\alpha, \beta)$ (see [11]). For every two desired vertices α, β of $V(A)$ we indicate $n^A_\alpha(f) = |W^A_{a,\beta}| = |\{a \in V(A) | d_A(a, \alpha) < d_A(a, \beta)\}|$ (see [1, 5]). In the same way, we would define $n^A_\beta(f) = |W^A_{\beta,\alpha}|$. We name A distance-balanced (DB) while for adjacent vertices α and β of A, we have $|W^A_{\alpha,\beta}| = |W^A_{\beta,\alpha}|$.

For each two desired edges $f = \alpha\beta$, $\acute{f} = \acute{\alpha}\acute{\beta}$, the distance between f and $\acute{f}$ is introduced via:

$$d_A(f, \acute{f}) = \min\{d_A(\alpha, \acute{f}), d_A(\beta, \acute{f})\} = \min\{d_A(\alpha, \acute{\alpha}), d_A(\alpha, \acute{\beta}), d_A(\beta, \acute{\alpha}), d_A(\beta, \acute{\beta})\}.$$

[1]speaker

Set $M_\alpha^A(f) = \{\acute{f} \in E(A) | d_A(\alpha, \acute{f}) < d_A(\beta, \acute{f})\}$ and $m_\alpha^A(f) = |M_\alpha^A(f)|$,
$M_\beta^A(f) = \{\acute{f} \in E(A) | d_A(\beta, \acute{f}) < d_A(\alpha, \acute{f})\}$ and $m_\beta^A(f) = |M_\beta^A(f)|$,
and $M_0^A(f) = \{\acute{f} \in E(A) | d_A(\alpha, \acute{f}) = d_A(\beta, \acute{f})\}$ and $m_0^A(f) = |M_0^A(f)|$.

Persume that $f = \alpha\beta \in E(A)$. For every two integers i, j we consider:

$$\acute{D}_j^i(f) = \{\acute{f} \in E(A) | d_A(\acute{f}, \alpha) = i, d_A(\acute{f}, \beta) = j\}.$$

A "distance partition" of $E(A)$ is concluded by sets $\acute{D}_j^i(f)$ duo to the edge $f = \alpha\beta$. Only the sets $\acute{D}_i^{i-1}(f)$, $\acute{D}_i^i(f)$ and, $\acute{D}_{i-1}^i(f)$, for each $(1 \leqslant i \leqslant d)$ might be nonempty according to the triangle inequality (The diameter of the graph A is d), as well as $\acute{D}_0^0(f) = \phi$.

A graph A is called *edge distance-balanced* (briefly EDB), while we have $m_g^A(f) = m_h^A(f)$. Also We call *A strongly edge distance-balanced* ($SEDB$), whenever for each edge $f = \alpha\beta$ and $i \geq 1$ it holds:

$$\acute{D}_{i-1}^i(f) = \acute{D}_i^{i-1}(f)$$

2 $SEDB$ property

In this segment, we present a characterization of $SEDB$ graphs and demonstrate that every edge-transitive graph is $SEDB$. Bring to mind, a graph A is said to be edge-transitive, if automorphism A acts on $E(A)$ transitively. In a graph A for an edge $f = \alpha\beta$ and each integer $i \geqslant 0$, consider $\acute{S}_i(f_\alpha)$ and $\acute{S}_i(f_\beta)$ denote all the edges at distance i to α and β, respectively, that is $\acute{S}_i(f_\alpha) = \{\acute{f} = \acute{\alpha}\acute{\beta} \in E(A) | d(\acute{f}, \alpha) = i\}$ and $\acute{S}_i(f_\beta) = \{\acute{f} = \acute{\alpha}\acute{\beta} \in E(A) | d(\acute{f}, \beta) = i\}$.

Proposition 2.1. *Suppose that A is a graph with diameter d. Then for each edge $f = \alpha\beta \in E(A)$ and each $i \in \{0, 1, ..., d\}$, A is $SEDB$ if and only if $|\acute{S}_i(f_\alpha)| = |\acute{S}_i(f_\beta)|$.*

Proof. Persume first that A is $SEDB$ and $f = \alpha\beta \in E(A)$. By definititon, we have $|\acute{D}_{i+1}^i(f)| = |\acute{D}_i^{i+1}(f)|$ for $i \in \{0, 1, ..., d-1\}$. Observe that $\acute{S}_i(f_\alpha)$ is a disjoin union of the sets $\acute{D}_{i-1}^i(f)$, $\acute{D}_i^i(f)$ and $\acute{D}_i^{i+1}(f)$. Analogously, $\acute{S}_i(f_\beta)$ is a disjoint union of the sets $\acute{D}_{i-1}^i(f)$, $\acute{D}_i^i(f)$ and $\acute{D}_i^{i+1}(f)$. Since $\acute{S}_i(f_\alpha) = \acute{D}_{i-1}^i(f) \cup \acute{D}_i^i(f) \cup \acute{D}_{i+1}^i(f)$ and $\acute{S}_i(f_\beta) = \acute{D}_i^{i-1}(f) \cup \acute{D}_i^i(f) \cup \acute{D}_i^{i+1}(f)$, we obtain $|\acute{S}_i(f_\alpha)| = |\acute{S}_i(f_\beta)|$.
For converse, consider that $|\acute{S}_i(f_\alpha)| = |\acute{S}_i(f_\beta)|$, for each edge $f = \alpha\beta \in E(A)$ and for some $0 \leqslant i \leqslant d-1$. By induction we have $|\acute{D}_{i+1}^i(f)| = |\acute{D}_i^{i+1}(f)|$, for each edge $f = \alpha\beta \in E(A)$ and for some $0 \leqslant i \leqslant d-1$. Assume now that $|\acute{D}_j^{j-1}(f)| = |\acute{D}_{j-1}^j(f)|$ holds for each $j \in \{1, ..., d-1\}$. We attain that

$$|\acute{D}_{j+1}^j(f)| = |\acute{S}_j(f_\alpha)| - |\acute{D}_j^j(f)| - |\acute{D}_{j-1}^j(f)|$$

and

$$|\acute{D}_j^{j+1}(f)| = |\acute{S}_j(f_\beta)| - |\acute{D}_j^j(f)| - |\acute{D}_j^{j-1}(f)|.$$

Since $|\acute{S}_j(f_\alpha)| = |\acute{S}_j(f_\beta)|$ and according to the induction hyphothesis $|\acute{D}_j^{j-1}(f)| = |\acute{D}_{j-1}^j(f)|$, we have $|\acute{D}_{j+1}^j(f)| = |\acute{D}_j^{j+1}(f)|$. The proof is completed. □

Assume that A is a $SEDB$ graph with diameter d. By Proposition 2.1 for every edge $f = \alpha\beta \in E(A)$ and for some $0 \leqslant i \leqslant d-1$, $|\acute{S}_i(f_\alpha)| = |\acute{S}_i(f_\beta)|$ holds. If automorphisms preserve distance, then the following corollary follows.

Corollary 2.2. *Let A be an edge-transitive graph. Then A is $SEDB$.*

3 $SEDB$ property in graph products

We would now investigate situations in which the *Cartesian product* leads to a $SNEDB$ graph. We mention that such product graphs, formed by graphs A and B, its vertex set is $V(A\Box B) = V(A) \times V(B)$. Consider that (a_1, b_1) and (a_2, b_2) are detached vertices in $V(A\Box B)$. In the Cartesian product $A\Box B$, if vertices (a_1, b_1) and (a_2, b_2) are coincident in one coordinate and adjacent in the another coordinate, then they are adjacent, that is, $a_1 = a_2$ and $b_1b_2 \in E(B)$, or $b_1 = b_2$ and $a_1a_2 \in E(A)$. Obviously, for vertices we have:

$$d_{A\Box B}((a_1, b_1), (a_2, b_2)) = d_A(a_1, a_2) + d_B(b_1, b_2).$$

For edges we have:

$$d_{A\Box B}((a, b)(a_1, b_1), (\acute{a}, \acute{b})(\acute{a}_1, \acute{b}_1)) =$$

$$\min\{d_{A\Box B}((a, b), (\acute{a}, \acute{b})), d_{A\Box B}((a, b), (\acute{a}_1, \acute{b}_1)), d_{A\Box B}((a_1, b_1), (\acute{a}, \acute{b})), \\ d_{A\Box B}((a_1, b_1), (\acute{a}_1, \acute{b}_1))\} =$$

$$\min\{d_A(a, \acute{a}) + d_B(b, \acute{b}), d_A(a, \acute{a}_1) + d_B(b, \acute{b}_1), d_A(a_1, \acute{a}) + d_B(b_1, \acute{b}), d_A(a_1, \acute{a}_1) + d_B(b_1, \acute{b}_1)\}.$$

Theorem 3.1. *Let A and B be connected graphs. Then $A\Box B$ is $SEDB$ if and only if both A and B are $SEDB$.*

Proof. Consider that $F = (a, b)(\acute{a}, \acute{b}) \in E(A\Box B)$ and $f = a\acute{a} \in E(A)$ and $\acute{f} = b\acute{b} \in E(B)$ and let $i \geqslant 0$. Then

$$\acute{S}_i(F_{(a,b)}) = \textstyle\bigcup_{j=0}^{i} \acute{S}_j(f_a) \times \acute{S}_{i-j}(\acute{f}_b)$$

$$\acute{S}_i(F_{(\acute{a},\acute{b})}) = \textstyle\bigcup_{j=0}^{i} \acute{S}_j(f_{\acute{a}}) \times \acute{S}_{i-j}(\acute{f}_{\acute{b}})$$

and therefore, recpectively

$$|\acute{S}_i(F_{(a,b)})| = \textstyle\sum_{j=0}^{i} |\acute{S}_j(f_a)||\acute{S}_{i-j}(\acute{f}_b)| \quad (1)$$

$$|\acute{S}_i(F_{(\acute{a},\acute{b})})| = \textstyle\sum_{j=0}^{i} |\acute{S}_j(f_{\acute{a}})||\acute{S}_{i-j}(\acute{f}_{\acute{b}})|. \quad (2)$$

Persume first that A and B are $SEDB$. Then, by Proposition 2.1 the number of edges of graph A and graph B at distance j from f_a and $\acute{f}_b$, respectively depend only on j. Hence by (1), (2) the number of edges of $A\Box B$ at distance i from $F = (a, b)(\acute{a}, \acute{b})$ depends on i, yielding that $A\Box B$ is $SEDB$.
For converse, suppose that A or B is not $SEDB$. We introduce r_C to be ∞ for a graph C if C is $SEDB$ and otherwise

$$\min\{i \in \mathbb{Z}| \text{ there exist } C_1 = x_1y_1, C_2 = x_2y_2 \in E(C) \text{ such that } |\acute{S}_i(C_{1x_1})| \neq \acute{S}_i(C_{2x_2})| \text{ and} \\ |\acute{S}_i(C_{1y_1})| \neq \acute{S}_i(C_{2y_2})|\}.$$

Assume that $i = \min\{r_A, r_B\}$ and it is clear that $i \leqslant \infty$. Now let $i = r_A$. Let $f_1 = a_1a_2$, $\acute{f}_1 = \acute{a}_1\acute{a}_2 \in E(A)$ such that $|\acute{S}_i(f_{1a_1})| > |\acute{S}_i(\acute{f}_{1\acute{a_1}})|$ and $|\acute{S}_i(f_{1a_2})| > |\acute{S}_i(\acute{f}_{1\acute{a_2}})|$ and let $f_2 = b_1b_2 \in E(B)$. By (1), (2) and the assumption A is not $SEDB$. For $(a_1, b_1)(a_2b_2) = F_1 \in E(A\Box B)$ and $(\acute{a}_1, b_1)(\acute{a}_2b_2) = F_1 \in E(A\Box B)$ we attain

$$\{|\acute{S}_i(F_{1(a_1,b_1)})|, |\acute{S}_i(F_{1(a_2,b_2)})|\} - \min\{|\acute{S}_i(\acute{F}_{1(\acute{a}_1,\acute{b}_1)})|, |\acute{S}_i(\acute{F}_{1(\acute{a}_2,\acute{b}_2)})|\} =$$

$$\min\{|\acute{S}_i(f_{1a_1})||\acute{S}_0(f_{2b_1})|, |\acute{S}_i(f_{1a_1})||\acute{S}_0(f_{2b_2})|,$$

$$|\acute{S}_i(f_{1a_2})||\acute{S}_0(f_{2b_1})|, |\acute{S}_i(f_{1a_2})||\acute{S}_0(f_{2b_2})|\}-$$

$$\min\{|\acute{S}_i(\acute{f}_{1\acute{a}_1})||\acute{S}_0(f_{2b_1})|, |\acute{S}_i(\acute{f}_{1\acute{a}_1})||\acute{S}_0(f_{2b_2})|,$$

$$|\acute{S}_i(\acute{f}_{1\acute{a}_2})||\acute{S}_0(f_{2b_1})|, |\acute{S}_i(\acute{f}_{1\acute{a}_2})||\acute{S}_0(f_{2b_2})|\} =$$

$$\min\{|\acute{S}_i(f_{1a_1})|, |\acute{S}_i(f_{1a_2})|\} - \min\{|\acute{S}_i(\acute{f}_{1\acute{a}_1})|, |\acute{S}_i(\acute{f}_{1\acute{a}_2})|\} > 0.$$

Thus, $A\square B$ is not $SEDB$. □

We would define the lexicographic product graphs. Product graph $A[B]$ of the graphs A and B, where its vertex set is $V(A[B]) = V(A) \times V(B)$, and two its adjacent vertices are $(a_1, b_1), (a_2, b_2)$ is defined the *lexicographic product* if $a_1a_2 \in E(A)$ or if $a_1 = a_2$ and also $b_1b_2 \in E(B)$ (for more information see [7, p. 22]). Since A is a graph, thus it is easily seen for vertices that

$$d_{A[B]}((a_1, b_1), (a_2, b_2)) = \begin{cases} d_A(a_1, a_2) & if a_1 \neq a_2 \\ \min\{2, d_B(b_1, b_2)\} & if a_1 = a_2. \end{cases}$$

And for edges we have:

$$d_{A[B]}((a, b)(a_1, b_1), (\acute{a}, \acute{b})(\acute{a}_1, \acute{b}_1)) =$$

$$\min \left\{ \begin{array}{cccc} d_A(a, \acute{a}) & if a \neq \acute{a}, & \min\{2, d_B(b, \acute{b})\} & if a = \acute{a} \\ d_A(a, \acute{a_1}) & if a \neq \acute{a_1}, & \min\{2, d_B(b, \acute{b_1})\} & if a = \acute{a_1} \\ d_A(a_1, \acute{a}) & if a_1 \neq \acute{a}, & \min\{2, d_B(b_1, \acute{b})\} & if a_1 = \acute{a} \\ d_A(a_1, \acute{a}_1) & if a_1 \neq \acute{a}_1, & \min\{2, d_B(b_1, \acute{b}_1)\} & if a_1 = \acute{a}_1 \end{array} \right\}.$$

Theorem 3.2. *Persume that A and B are graphs and $A[B]$ is connected. Then, $A[B]$ is $SEDB$ if and only if A is $SEDB$ and also B is regular.*

Proof. Let $F = (a, b)(\acute{a}, \acute{b}) \in E(A[B])$ and d be the diameter of $A[B]$. It is clearly seen that since $A[B]$ is connected also A is connected and for $f = a\acute{a} \in E(A)$ and $\acute{f} = b\acute{b} \in E(B)$ we have

$$\acute{S}_0(F_{(a,b)}) = \acute{S}_0(f_a) \times E(B) \cup \{(a, b_1)(\acute{a}, b_2)|(b_1, b_2) \in \acute{S}_0(\acute{f}_b)\}$$

and also

$$\acute{S}_0(F_{(\acute{a},\acute{b})}) = \acute{S}_0(f_{\acute{a}}) \times E(B) \cup \{(a, b_1)(\acute{a}, b_2)|(b_1, b_2) \in \acute{S}_0(\acute{f}_{\acute{b}})\}.$$

Therefore

$$\acute{S}_1(F_{(a,b)}) = \acute{S}_1(f_a) \times E(B) \cup \{(a, b_1)(\acute{a}, b_2)|(b_1, b_2) \notin \acute{S}_1(\acute{f}_b)\}$$

$$\acute{S}_1(F_{(\acute{a},\acute{b})}) = \acute{S}_1(f_{\acute{a}}) \times E(B) \cup \{(a, b_1)(\acute{a}, b_2)|(b_1, b_2) \notin \acute{S}_1(\acute{f}_{\acute{b}})\},$$

$$\acute{S}_i(F_{(a,b)}) = \acute{S}_i(f_a) \times E(B) \qquad i \in \{2, 3, ..., d\} \tag{3}$$

$$\acute{S}_i(F_{(\acute{a},\acute{b})}) = \acute{S}_i(f_{\acute{a}}) \times E(B) \qquad i \in \{2, 3, ..., d\}. \tag{4}$$

Consider first that A is $SEDB$ and B is regular. By (3), (4)

$$\min\{|\acute{S}_i(a_1, \acute{b}_1)(a_2, \acute{b}_2)_{(a_1,\acute{b}_1)}|, |\acute{S}_i(a_1, \acute{b}_1)(a_2, \acute{b}_2)_{(a_2,\acute{b}_2)}|\}=$$

$$\min\{|\acute{S}_i(\acute{a}_1,\acute{\acute{b}}_1)(\acute{a}_2,\acute{\acute{b}}_2)_{(\acute{a}_1,\acute{\acute{b}}_1)}|,|\acute{S}_i(\acute{a}_1,\acute{\acute{b}}_1)(\acute{a}_2,\acute{\acute{b}}_2)_{(\acute{a}_2,\acute{\acute{b}}_2)}|\},$$

for every two edges $(a_1,\acute{b}_1)$ and $(\acute{a}_1,\acute{\acute{b}}_1)(\acute{a}_2,\acute{\acute{b}}_2)$ of $A[B]$ and for every integer i. Consequently, by Proposition 2.1 $A[B]$ is $SEDB$. Suppose next that $A[B]$ is $SEDB$. Let $a\acute{a} \in E(A)$ and $\acute{b}_1\acute{b}_2, \acute{\acute{b}}_1\acute{\acute{b}}_2 \in E(B)$. Then,

$$\min\{|\acute{S}_0(a,\acute{b}_1)(\acute{a},\acute{b}_2)_{(a,\acute{b}_1)}|,|\acute{S}_0(a,\acute{b}_1)(\acute{a},\acute{b}_2)_{(\acute{a},\acute{b}_2)}|\}=$$

$$\min\{|\acute{S}_0(a,\acute{\acute{b}}_1)(\acute{a},\acute{\acute{b}}_2)_{(a,\acute{\acute{b}}_1)}|,|\acute{S}_0(a,\acute{\acute{b}}_1)(\acute{a},\acute{\acute{b}}_2)_{(\acute{a},\acute{\acute{b}}_2)}|\},$$

and (1), (2) imply that

$$\min\{|\acute{S}_0(\acute{b}_1\acute{b}_2)_{\acute{b}_1}|,|\acute{S}_0(\acute{b}_1\acute{b}_2)_{\acute{b}_2}|\}= \min\{|\acute{S}_0(\acute{\acute{b}}_1\acute{\acute{b}}_2)_{\acute{\acute{b}}_1}|,|\acute{S}_0(\acute{\acute{b}}_1\acute{\acute{b}}_2)_{\acute{\acute{b}}_2}|\}.$$

Thus, B is regular. Considering this and by (2), (3), $|\acute{S}_i(a\acute{a})_a|$ and $|\acute{S}_i(a\acute{a})_{\acute{a}}|$ depend only on i, yielding that A is $SEDB$. □

References

[1] A. Abedi, M. Alaeiyan, A. Hujdurović and K. Kutnar, *Quasi-λ-distance-balanced graphs*, Discrete Appl. Math., 227 (2017), pp. 21-28.

[2] Z. Aliannejadi, A. Gilani, M. Alaeiyan and J. Asadpour, *On some properties of edge quasi-distance-balanced graphs*, Journal of Mathematical Extension. 16 (2022), pp. 1-13.

[3] Z. Aliannejadi, A. Gilani, M. Alaeiyan and J. Asadpour, *2-edge distance-balanced graphs*, Journal of Positive School Psychology, 6 (2022), pp. 10058-10064.

[4] K. Balakrishnan, M. Changat, I. Peterin, S. Špacapan, P. Šparl and A. R. Subhamathi, *Strongly distance-balanced graphs and graph products*, European J. Combin., 30 (2009), pp: 1048-1053.

[5] A. Ilič, S. Klavžar and M. Milanović, *On distance-balanced graphs*, European J. Combin., 31 (2010), pp: 733-737.

[6] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, Wiley, New York, USA, 2000.

[7] J. Jerebic, S. Klavžar and D.F. Rall, *Distance-balanced graphs*, Ann. Comb., 12 (2008), pp: 71-79.

[8] K. Kutnar, A. Malnič, D. Marušič and Š. Miklavič, *Distance-balanced graphs: Symmetry conditions*, Discrete Math., 306 (2006), pp: 1881-1894.

[9] K. Kutnar and Š. Miklavič, *Nicely distance-balanced graphs*, European J. Combin., 39 (2014), pp: 57-67.

[10] Š. Miklavič and P. Šparl, *On the connectivity of bipartite distance-balanced graphs*, European J. Combin., 33, (2012), pp: 237-247.

[11] M. Tavakoli, H. Yousefi-azari and A.R. Ashrafi, *Note on Edge Distance-Balanced Graphs*, Transaction on Combinatorics, University of Isfehan, 1, no. 1 (2012), pp: 1-6.

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