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Perfect 3-Coloring and Truncated Octahedral Graph

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Abstract

If the vertex set of the graph G can be partitioned into m parts $V_1, V_2, \dots, V_m$ in such a way that the number of neighbors of a vertex in each of these parts is a constant number depends on the parts, that is the number of neighbor of a vertex $v \in V_i$ in V_j is a_{ij}, where $1 \leq i, j \leq n$, this partition is a perfect m-coloring for G and is perfect m-colorable. Also, the $m \times m$ matrix $A = (a_{ij})$ is called parameter matrix of the graph. In this paper, we find all possible parameter matrices of perfect 3-coloring for truncated octahedral graph.

Keywords: Vertex coloring, Perfect 3-Coloring, Parameter matrix, Truncated octahedral graph.

AMS Mathematical Subject Classification [2010]: 05C15

1 Introduction

This note is about perfect 3-coloring of truncated octahedral graph. Like other kinds of coloring, this concept of graph theory related to a partition of vertices that has special property, too. Our definitions and terminologies are from [11]. Let $G = (V, E)$ be a graph. A partition $(V_1, V_2, \dots, V_m)$ of V is *perfect m-coloring* of G, if for every i, j, $1 \leq i, j \leq m$, each vertex $v \in V_i$ has exactly a_{ij} neighbors in V_j. The matrix $A = (a_{ij})_{1 \leq i,j \leq m}$ is called the *parameter matrix*. For more information see [1–7, 9, 10].

Two parameter matrices A and A' are *isomorphic* if a permutation $\sigma \in S_m$ exists that operates on both rows and columns of A to obtain A'. It is obvious that if A and A' are isomorphic, then the existence of a coloring with parameter matrix A is equivalent the existence of a coloring with parameter matrix A'.

Let $G = (V, E)$ be a graph. Suppose that M be a matching in graph $G = (V, E)$. If $G[M] = M$, the we call M an *induced matching*. Suppose that C be a union of cycles in graph $G = (V, E)$. If $G[C] = C$, the we call C an *induced cycles*.

[1]speaker

In [1] we determined all parameter matrices of perfect 3-colorings of cubic graphs and properties of perfect 3-colorable cubic graph with each parameter matrix, (see Table 1). Also we characterize perfect 3-colorable cubic graphs, (see Lemma 2.1). In this paper, we determine all perfect 3-coloring for truncated octahedral graph:

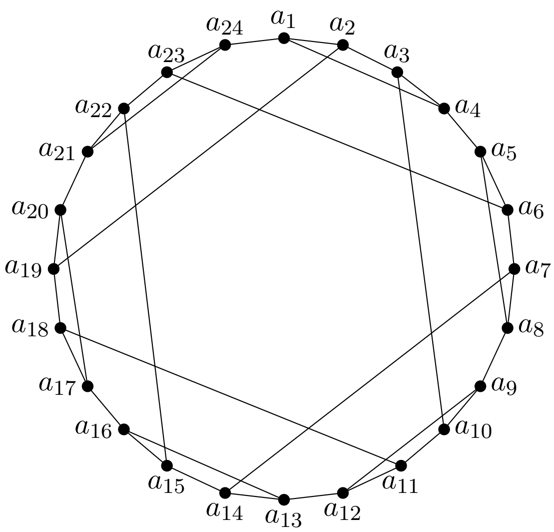

Figure 1: Truncated Octahedral Graph

Throughout this paper, our focus is on perfect 3-colorings, that is shown by (W, B, R), where $|W| \leq |B| \leq |R|$, and W, B and R are symbols for white, black and red, respectively.

2 Preliminaries

In this section, we introduced some preliminary result that will be used to solve the problem. The first one is Table 1 (see [1]) in which you can see the properties of the perfect 3-coloring of cubic graphs of order n, with determined parameter matrix. All parameter matrices of the perfect 3-coloring of cubic graphs of order n which is determined in [1] are:

$$A_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad A_7 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad A_8 = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad A_9 = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad A_{10} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \quad A_{13} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad A_{14} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \quad A_{15} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A_{16} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad A_{17} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad A_{18} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

In each row, the 1st box is the parameter matrix. The 2nd, 3rd and 4th one are the size of coloring parts W, B and R, respectively. The 5th, 6th and 7th one show that coloring parts W, B and R are independent, induced matching or induced cycles, respectively.

The second one is Lemma 2.1 which characterize the cubic graph that has perfect 3-coloring.

Table 1: Complete properties of all parameter matrices

M	$\|W\|$	$\|B\|$	$\|R\|$	W	B	R	M	$\|W\|$	$\|B\|$	$\|R\|$	W	B	R
A_1	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	I	C	M	A_2	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{2}$	I	C	C
A_3	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	M	C	C	A_4	$\frac{n}{7}$	$\frac{3n}{7}$	$\frac{3n}{7}$	I	M	C
A_5	$\frac{n}{4}$	$\frac{3n}{8}$	$\frac{3n}{8}$	I	I	C	A_6	$\frac{n}{10}$	$\frac{3n}{10}$	$\frac{3n}{5}$	I	I	C
A_7	$\frac{n}{7}$	$\frac{3n}{7}$	$\frac{3n}{7}$	I	I	M	A_8	$\frac{n}{5}$	$\frac{2n}{5}$	$\frac{2n}{5}$	M	M	C
A_9	$\frac{n}{7}$	$\frac{2n}{7}$	$\frac{4n}{7}$	M	I	C	A_{10}	$\frac{n}{5}$	$\frac{2n}{5}$	$\frac{2n}{5}$	M	I	M
A_{11}	$\frac{n}{5}$	$\frac{n}{5}$	$\frac{3n}{5}$	I	I	M	A_{12}	$\frac{n}{6}$	$\frac{n}{3}$	$\frac{n}{2}$	I	I	I
A_{13}	$\frac{2n}{11}$	$\frac{3n}{11}$	$\frac{6n}{11}$	I	M	M	A_{14}	$\frac{3n}{13}$	$\frac{4n}{13}$	$\frac{6n}{13}$	M	I	I
A_{15}	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{2}$	M	M	M	A_{16}	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	I	M	I
A_{17}	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{2}$	I	I	M	A_{18}	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	M	M	M

Lemma 2.1. *[1] If the 3-regular graph G of order n has perfect 3-coloring with parameter matrix*

1. *A_1, A_3, A_{12}, A_{16} or A_{18}, then $6|n$.*
2. *A_2 or A_{17}, then $4|n$.*
3. *A_4, A_7 or A_9, then $14|n$.*
4. *A_5 and A_{15} then $8|n$.*
5. *A_6, A_8, A_{10} or A_{11} then $10|n$.*
6. *A_{13} then $22|n$.*
7. *A_{14} then $26|n$.*

The third one is Lemma 2.2 in which the relation between the spectra of parameter matrices and the spectra of the graph determined

Lemma 2.2. *[8] If A is a perfect m-coloring matrix of a graph G, then any eigenvalue of A is an eigenvalue of G.*

The last one is the spectrum of all possible parameter matrices for a perfect 3-coloring and Truncated octahedral graph graph.

Lemma 2.3. *[12] Truncated octahedral graph graph spectrum is*

$$(\pm 3)^1(\pm 1)^3(\pm 1 \pm \sqrt{2})^3(\pm\sqrt{3})^2$$

Table 2: Spectrum of parameter matrices.

Matrix	λ_1	λ_2	λ_3	Matrix	λ_1	λ_2	λ_3	Matrix	λ_1	λ_2	λ_3
A_1	3	$\sqrt{3}$	$-\sqrt{3}$	A_2	3	2	-1	A_3	3	2	0
A_4	3	$\sqrt{2}$	$-\sqrt{2}$	A_5	3	$\frac{-1+\sqrt{17}}{2}$	$\frac{-1-\sqrt{17}}{2}$	A_6	3	1	-2
A_7	3	$-1+\sqrt{2}$	$-1-\sqrt{2}$	A_8	3	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	A_9	3	$\sqrt{2}$	$-\sqrt{2}$
A_{10}	3	1	-2	A_{11}	3	0	-2	A_{12}	3	0	-3
A_{13}	3	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	A_{14}	3	$-1+\sqrt{3}$	$-1-\sqrt{3}$	A_{15}	3	1	-1
A_{16}	3	0	-2	A_{17}	3	-1	-1	A_{18}	3	0	0

3 main results

Theorem 3.1. *There is a perfect 3-coloring for truncated octahedral graph only with parameter matrix A_1.*

Proof: You can see perfect 3-coloring with each parameter matrices A_1 in Table 4.
By Lemmas 2.2 and 2.3 and Table 2, A_2, A_3, A_4, A_5, A_6, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{16} and A_{18} are not parameter matrices.
Part 3 of Lemma 2.1 implies that A_7 is not a parameter matrix, too.
Let (W, B, R) be a perfect 3-coloring with parameter matrix A_{15}. By the graph construction, it can be suppose that $a_1a_4 \in R$ or $a_1a_{24} \in R$. In the former case, w.l.o.g. we can assume that $a_2 \in W$ and $a_{24} \in B$. Therefore $a_3 \in W$ and $a_5 \in B$. Since two member of B (or W) do not have a common neighbor, so $a_{23} \in R$ and $a_6 \in R$. Now, $a_7 \in W$ and $a_8 \in B$, that is impossible. In the later one, $a_1a_{24} \in R$, we can assume that $a_2 \in B$ and $a_4 \in W$. Therefore $a_3, a_{24} \in R$, $a_5 \in W$ and $a_{23} \in B$. Now, $a_7, a_8 \in R$, that is impossible too and there is no perfect 3-coloring with parameter matrix A_{15}. By similar method there is no perfect 3-coloring with parameter matrix A_{17}.

Table 3: A Coloring for the Parameter Matrix A_1

A_1	W	$a_1, a_3, a_6, a_8, a_{13}, a_{15}, a_{18}, a_{20}$
	B	$a_9, a_{10}, a_{11}, a_{12}, a_{21}, a_{22}, a_{23}, a_{24}$
	R	$a_2, a_4, a_5, a_7, a_{14}, a_{16}, a_{17}, a_{19}$

Theorem 3.2. *There is a perfect 3-coloring for truncated octahedral graph only with parameter matrix A_1.*

Proof: You can see perfect 3-coloring with each parameter matrices A_1 in Table 4.
By Lemmas 2.2 and 2.3 and Table 2, A_2, A_3, A_4, A_5, A_6, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{16} and A_{18} are not parameter matrices.
Part 3 of Lemma 2.1 implies that A_7 is not a parameter matrix, too.
Let (W, B, R) be a perfect 3-coloring with parameter matrix A_{15}. By the graph construction,

it can be suppose that $a_1a_4 \in R$ or $a_1a_{24} \in R$. In the former case, w.l.o.g. we can assume that $a_2 \in W$ and $a_{24} \in B$. Therefore $a_3 \in W$ and $a_5 \in B$. Since two member of B (or W) do not have a common neighbor, so $a_{23} \in R$ and $a_6 \in R$. Now, $a_7 \in W$ and $a_8 \in B$, that is impossible. In the later one, $a_1a_{24} \in R$, we can assume that $a_2 \in B$ and $a_4 \in W$. Therefore $a_3, a_{24} \in R$, $a_5 \in W$ and $a_{23} \in B$. Now, $a_7, a_8 \in R$, that is impossible too and there is no perfect 3-coloring with parameter matrix A_{15}. By similar method there is no perfect 3-coloring with parameter matrix A_{17}.

Table 4: A Coloring for the Parameter Matrix A_1

A_1	W	$a_1, a_3, a_6, a_8, a_{13}, a_{15}, a_{18}, a_{20}$
	B	$a_9, a_{10}, a_{11}, a_{12}, a_{21}, a_{22}, a_{23}, a_{24}$
	R	$a_2, a_4, a_5, a_7, a_{14}, a_{16}, a_{17}, a_{19}$

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