7th International Conference on
Combinatorics, Cryptography, Computer Science and Computing
November. 16-17, 2022

Perfect 3-Coloring and Truncated Cubical Graph

Mehdi Alaeiyan[1]
School of Mathematics, Iran University of Science and Technology, Narmak, Tehran.

Sahar Qajar
Department of Mathematical Science, Sharif University of Technology,
Azadi Street, P. O. Box 11365-9415, Tehran, Iran.

Abstract

If the vertex set of the graph G can be partitioned into m parts $V_1, V_2, \ldots, V_m$ in such a way that the number of neighbors of a vertex in each of these parts is a constant number depends on the parts, that is the number of neighbor of a vertex $v \in V_i$ in V_j is a_{ij}, where $1 \leq i, j \leq n$, this partition is a perfect m-coloring for G and is perfect m-colorable. Also, the $m \times m$ matrix $A = (a_{ij})$ is called parameter matrix of the graph. In this paper, we find all possible parameter matrices of perfect 3-coloring for truncated cubical graph.

Keywords: Vertex coloring, Perfect 3-Coloring, Parameter matrix, Truncated cubical graph
AMS Mathematical Subject Classification [2010]: 05C15

1 Introduction

This note is about perfect 3-coloring of truncated cubical graph. Like other kinds of coloring, this concept of graph theory related to a partition of vertices that has special property, too. Our definitions and terminologies are from [11]. Let $G = (V, E)$ be a graph. A partition $(V_1, V_2, \ldots, V_m)$ of V is *perfect m-coloring* of G, if for every i, j, $1 \leq i, j \leq m$, each vertex $v \in V_i$ has exactly a_{ij} neighbors in V_j. The matrix $A = (a_{ij})_{1 \leq i,j \leq m}$ is called the *parameter matrix*. For more information see [1–7, 9, 10].

Two parameter matrices A and A' are *isomorphic* if a permutation $\sigma \in S_m$ exists that operates on both rows and columns of A to obtain A'. It is obvious that if A and A' are isomorphic, then the existence of a coloring with parameter matrix A is equivalent the existence of a coloring with parameter matrix A'.

Let $G = (V, E)$ be a graph. Suppose that M be a matching in graph $G = (V, E)$. If $G[M] = M$, the we call M an *induced matching*. Suppose that C be a union of cycles in graph $G = (V, E)$. If $G[C] = C$, the we call C an *induced cycles*.

In [1] we determined all parameter matrices of perfect 3-colorings of cubic graphs and properties of perfect 3-colorable cubic graph with each parameter matrix, (see Table 1). Also we characterize perfect 3-colorable cubic graphs, (see Lemma 2.1).

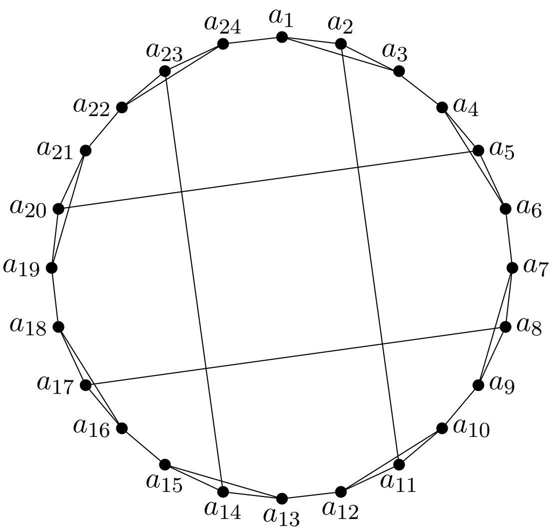

Figure 1: Truncated Cubical Graph

In this paper we determine all perfect 3-coloring for truncated cubical graph:

Throughout this paper, our focus is on perfect 3-colorings, that is shown by (W, B, R), where $|W| \leq |B| \leq |R|$, and W, B and R are symbols for white, black and red, respectively.

2 Preliminaries

In this section, we introduced some preliminary result that will be used to solve the problem. The first one is Table 1 (see [1]) in which you can see the properties of the perfect 3-coloring of cubic graphs of order n, with determined parameter matrix. All parameter matrices of the perfect 3-coloring of cubic graphs of order n which is determined in [1] are:

$$A_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad A_7 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad A_8 = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad A_9 = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad A_{10} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \quad A_{13} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad A_{14} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \quad A_{15} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A_{16} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad A_{17} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad A_{18} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

[1]speaker

In each row, the 1st box is the parameter matrix. The 2nd, 3rd and 4th one are the size of coloring parts W, B and R, respectively. The 5th, 6th and 7th one show that coloring parts W, B and R are independent, induced matching or induced cycles, respectively.

The second one is Lemma 2.1 which characterize the cubic graph that has perfect 3-coloring.

Table 1: Complete properties of all parameter matrices

M	$\|W\|$	$\|B\|$	$\|R\|$	W	B	R	M	$\|W\|$	$\|B\|$	$\|R\|$	W	B	R
A_1	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	I	C	M	A_2	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{2}$	I	C	C
A_3	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	M	C	C	A_4	$\frac{n}{7}$	$\frac{3n}{7}$	$\frac{3n}{7}$	I	M	C
A_5	$\frac{n}{4}$	$\frac{3n}{8}$	$\frac{3n}{8}$	I	I	C	A_6	$\frac{n}{10}$	$\frac{3n}{10}$	$\frac{3n}{5}$	I	I	C
A_7	$\frac{n}{7}$	$\frac{3n}{7}$	$\frac{3n}{7}$	I	I	M	A_8	$\frac{n}{5}$	$\frac{2n}{5}$	$\frac{2n}{5}$	M	M	C
A_9	$\frac{n}{7}$	$\frac{2n}{7}$	$\frac{4n}{7}$	M	I	C	A_{10}	$\frac{n}{5}$	$\frac{2n}{5}$	$\frac{2n}{5}$	M	I	M
A_{11}	$\frac{n}{5}$	$\frac{n}{5}$	$\frac{3n}{5}$	I	I	M	A_{12}	$\frac{n}{6}$	$\frac{n}{3}$	$\frac{n}{2}$	I	I	I
A_{13}	$\frac{2n}{11}$	$\frac{3n}{11}$	$\frac{6n}{11}$	I	M	M	A_{14}	$\frac{3n}{13}$	$\frac{4n}{13}$	$\frac{6n}{13}$	M	I	I
A_{15}	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{2}$	M	M	M	A_{16}	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	I	M	I
A_{17}	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{2}$	I	I	M	A_{18}	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	M	M	M

Lemma 2.1. *[1] If the 3-regular graph G of order n has perfect 3-coloring with parameter matrix*

1. A_1, A_3, A_{12}, A_{16} *or* A_{18}, *then* $6|n$.
2. A_2 *or* A_{17}, *then* $4|n$.
3. A_4, A_7 *or* A_9, *then* $14|n$.
4. A_5 *and* A_{15} *then* $8|n$.
5. A_6, A_8, A_{10} *or* A_{11} *then* $10|n$.
6. A_{13} *then* $22|n$.
7. A_{14} *then* $26|n$.

The third one is Lemma 2.2 in which the relation between the spectra of parameter matrices and the spectra of the graph determined

Lemma 2.2. *[8] If A is a perfect m-coloring matrix of a graph G, then any eigenvalue of A is an eigenvalue of G.*

The last one is the spectrum of all possible parameter matrices for a perfect 3-coloring and Truncated cubical graph graph.

Lemma 2.3. *[12] Truncated cubical graph graph spectrum is*

$$(-2)^5(\frac{1}{2}(1-\sqrt{17}))^3(-1)^3 0^5 1^1 2^3(\frac{1}{2}(1+\sqrt{17}))^3 3^1$$

Table 2: Spectrum of parameter matrices.

Matrix	λ_1	λ_2	λ_3	Matrix	λ_1	λ_2	λ_3	Matrix	λ_1	λ_2	λ_3
A_1	3	$\sqrt{3}$	$-\sqrt{3}$	A_2	3	2	-1	A_3	3	2	0
A_4	3	$\sqrt{2}$	$-\sqrt{2}$	A_5	3	$\frac{-1+\sqrt{17}}{2}$	$\frac{-1-\sqrt{17}}{2}$	A_6	3	1	-2
A_7	3	$-1+\sqrt{2}$	$-1-\sqrt{2}$	A_8	3	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	A_9	3	$\sqrt{2}$	$-\sqrt{2}$
A_{10}	3	1	-2	A_{11}	3	0	-2	A_{12}	3	0	-3
A_{13}	3	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	A_{14}	3	$-1+\sqrt{3}$	$-1-\sqrt{3}$	A_{15}	3	1	-1
A_{16}	3	0	-2	A_{17}	3	-1	-1	A_{18}	3	0	0

3 main results

Theorem 3.1. *There is a perfect 3-coloring for truncated cubical graph only with parameter matrices A_2, A_{16}, A_{17} and A_{18}.*

Proof: You can see perfect 3-coloring with each parameter matrices A_2, A_{16}, A_{17} and A_{18} in Table 3.

By Lemmas 2.2 and 2.3 and Table 2, A_1, A_4, A_5, A_7, A_8, A_9, A_{12}, A_{13} and A_{14} are not parameter matrices.

Let (W, B, R) be a perfect 3-coloring with parameter matrix A_3. Since the graph is vertex transitive, we can assume that $a_1 \in W$. Since there is not any edges between B, it can be assume that $a_{24} \in B$ and so $a_{22}, a_{23} \in B$. Table 1 implies, B is an induced cycles of order 8, a contradiction.

Since the graph has order 24, by part 5 of Lemma 2.1 A_6, A_{10} and A_{11} are not parameter matrices.

For A_{15}, since there is no edges between W and B, we can have $a_1, a_{24} \in W$, $a_4, a_{11}, a_{14}, a_{21} \in B$ and $a_2, a_3, a_{22}, a_{23} \in R$. But B is an induced matching aof size 6 and this is a contradiction. Because $\{a_4, a_{11}, a_{14}, a_{21}\}$ is an independent set.

Table 3: A Coloring for the Parameter Matrices A_2, A_{16}, A_{17} and A_{18}

A_2	W	$a_4, a_8, a_{11}, a_{15}, a_{19}, a_{24}$
	B	$a_1, a_2, a_3, a_{16}, a_{17}, a_{18}$
	R	$a_5, a_6, a_7, a_9, a_{10}, a_{12}, a_{13}, a_{14}, a_{20}, a_{21}, a_{22}, a_{23}$
A_{16}	W	$a_3, a_6, a_9, a_{12}, a_{15}, a_{18}, a_{21}, a_{24}$
	B	$a_2, a_5, a_8, a_{11}, a_{14}, a_{17}, a_{20}, a_{23}$
	R	$a_1, a_4, a_7, a_{10}, a_{13}, a_{16}, a_{19}, a_{22}$
A_{17}	W	$a_1, a_6, a_{10}, a_{14}, a_{17}, a_{21}$
	B	$a_3, a_7, a_{12}, a_{16}, a_{20}, a_{23}$
	R	$a_2, a_4, a_5, a_8, a_9, a_{11}, a_{13}, a_{15}, a_{18}, a_{19}, a_{22}, a_{24}$
A_{18}	W	$a_1, a_6, a_7, a_{12}, a_{13}, a_{18}, a_{19}, a_{24}$
	B	$a_2, a_5, a_8, a_{11}, a_{14}, a_{17}, a_{20}, a_{23}$
	R	$a_3, a_4, a_9, a_{10}, a_{15}, a_{16}, a_{21}, a_{22}$

References

[1] M. Alaeiyan and S. Qajar, Parameter Matrices in Perfect 3-Coloring and Some Symmetric Cubic Graphs, in preparation.

[2] M. Alaeiyan and Zahra Shokoohi, Perfect 3-colorings of some generalized peterson graph, Commun. Combin., Cryptogr. and Computer Sci., 1 (2021), 136–142.

[3] M. Alaeiyan, Perfect 3-colorings Of Heawood Graph, Iran J Sci Technol Trans Sci,12 (2021) No. 1, 713-717. https://doi.org/10.1007/s40995-018-0646-1

[4] M. Alaeiyan and A. Mehrabani, Perfect 3-colorings of the Platonic graphs, Iran J Sci Technol Trans Sci https://doi.org/10.1007/s40995-018-0646-1

[5] M. Alaeiyan and A. Mehrabani, Perfect 3-colorings of the cubic graphs of order 10, Electron. J. Graph Theory Appl. 5 (2) (2017), 194–206.

[6] M Alaeiyan, H Karami, Perfect 2-colorings of the generalized Petersen graph, Proceedings-Mathematical Sciences 126, 289-294.

[7] S. V. Avgustinovich and I. Yu. Mogil'nykh, Perfect colorings of the Johnson graph $J(8, 3)$ and $J(8, 4)$ with two colors, Journal of Applied and Industrial Mathematics, 2011, Vol. 5, No. 1, pp. 19–30.

[8] C. Godsil, Compact graphs and equitable partitions, Linear Algebra and Its Application 255 (1997), 259–266.

[9] H. Karami, Perfect 2-colorings of the generalized Petersen graph $GP(n, 3)$, Arxiv,https://arxiv.org/abs/2009.07120.

[10] Z. Vahedi, M. Alaeiyan and M. Maghasedi, Perfect 4-colorings of the 3-regular graphs of order at most 8, Journal of Mathematical Extension, Journal Pre-proof, ISSN: 1735-8299 URL: http://www.ijmex.com.

[11] D. B. West, Introduction to graph theory, Prentice Hall 1996, 2001.

[12] https://mathworld.wolfram.com/TruncatedCubicalGraph.html.

e-mail: alaeiyan@iust.ac.ir
e-mail: s.qajar@sharif.edu