Self-complementary strongly regular Cayley graphs over abelian groups

Mojtaba Jazaeri[1]

Department of Mathematics, Faculty of Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran

Abstract

In this paper, we study self-complementary strongly regular Cayley graphs over abelian groups. We state two questions regarding this topic and give a partial answer to one of them.

Keywords: Self-complementary graph, Strongly regular graph, Cayley graph

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1 Introduction

A graph is called self-complementary if it is isomorphic to its complement. The structure of a self-complementary graph is beautiful and it has also application in finding lower bound in Ramsey numbers. It is well known that a self-complementary graph has diameter at most three with $4t + 1$ vertices, where t is a natural number. The classification of self-complementary regular graphs seems to be complicated and therefore the researchers have focused on some special families such as strongly regular graphs, Cayley graphs, vertex-transitive graphs, and symmetric graphs. In this paper, we concentrate on self-complementary strongly-regular Cayley graphs over abelian groups. In general, the study of strongly regular Cayley graphs are the same as partial difference sets (see [4]) and self-complementary strongly regular Cayley graphs come from Paley type partial difference sets. There exist two infinite families of self-complementary strongly regular Cayley graphs; Payley graphs and Peisert graphs. Let $\mathbb{F}$ be a finite field of order $q = p^r$, where p is a prime number and q congruent to 1 modulo 4. Then the Paley graph P_q is a Cayley graph $Cay(G, S)$ over the elementary abelian group G with the connection set $S = \{x^2 \mid x \in \mathbb{F}, x \neq 0\}$. If the multiplicative generator of the finite field $\mathbb{F}$ is denoted by a, where p is a prime number congruent to 3 modulo 4 and r is even, then the Peisert graph P^*_q is a Cayley graph $Cay(G, S)$ over the elementary abelian group G with the connection set $S = \{a^i \mid i \equiv 0, 1 \mod 4\}$. Similar to Arasu, Jungnickel, Ma and Pott questions in [1], we state the following questions.

Question 1.1. Let G be an abelian group of order $4t + 1$. If $4t + 1$ is not a prime power, does there exist a self-complementary strongly Cayley regular graph over the group G? If $4t + 1$ is a prime power, does the group G need to be elementary abelian?

[1]speaker

2 Main results

Let G be a finite group and S be an inverse-closed subset of G not containing the identity element; we call S the connection set. Then the Cayley graph $Cay(G, S)$ is the graph whose vertex set is G, where two vertices a and b are adjacent whenever $ab^{-1} \in S$. We note that the Cayley graph $Cay(G, S)$ is a regular graph of degree $|S|$ and it is connected if and only if the subgroup generated by the connection set S is equal to G. The complement of a graph Γ is denoted by Γ^c and the identity element of a group by e. We also note that the complement of the Cayley graph $\Gamma = Cay(G, S)$ is $\Gamma^c = Cay(G, G \setminus (S \cup \{e\}))$. It is well known that a strongly regular graph with n vertices and degree k has parameters (n, k, λ, μ) such that two adjacent vertices have λ common neighbors and two non-adjacent vertices has μ common neighbors.

In the next section, we give two methods to construct self-complementary Cayley graphs but these methods don't work to construct self-complementary strongly regular Cayley graphs in general.

2.1 Two methods for constructions of self-complementary Cayley graphs

Let Γ_1 and Γ_2 be two graphs with vertex set $V(\Gamma_1)$ and $V(\Gamma_1)$, respectively. Then the lexicographic product of the graph Γ_1 with Γ_2 which is denoted by $\Gamma_1[\Gamma_2]$, is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ such that two vertices (a, b) and (c, d), where $a, c \in V(\Gamma_1)$ and $b, d \in V(\Gamma_2)$, are adjacent whenever a is adjacent to c in Γ_1 or $a = c$ and b is adjacent to d in Γ_2. It is well known that the lexicographic product of a self-complementary graph Γ_1 with another self-complementary graph Γ_2 is a self-complementary graph. On the other hand, it is trivial to see that the lexicographic product of a Cayley graph $Cay(G_1, S_1)$ with another Cayley graph $Cay(G_2, S_2)$ is also the Cayley graph $Cay(G_1 \times G_2, S)$, where $S = \{(a, g) \mid a \in S_1, g \in G_2\} \cup \{(e, s) \mid s \in S_2\}$. This implies that the lexicographic product of a self-complementary Cayley graph with another self-complementary Cayley graph is a self-complementary Cayley graph.

Let G be a finite group and σ be an automorphism of the group G of order a power of 2 such that σ^2 is fixed-point free. Let $\Delta_1, \Delta_2, \ldots, \Delta_{2k}$ be the orbits of the action of $\langle\sigma^2\rangle$ on the set $\{\{g, g^{-1}\} \mid g \in G \setminus \{e\}\}$ such that

$$\sigma(\Delta_{2i-1}) = \Delta_{2i},$$

for $i = 1, 2, \ldots, k$ (cf. [8, §3]). Therefore the Cayley graph $Cay(G, S)$, where

$$S = \Delta_1 \cup \Delta_3 \cup \ldots \Delta_{2k-1},$$

is a self-complementary Cayley graph in such a way that

$$Cay(G, S) \cong Cay(G, \sigma(S)) \cong Cay(G, G \setminus (S \cup \{e\})).$$

2.2 Self-complementary strongly regular Cayley graphs over abelian groups

Polhill in [6] proved that there exist Paley type partial difference sets in the group $\mathbb{Z}_3^2 \times \mathbb{Z}_p^{4t}$, where t is a natural number and p is an odd prime number. Finally, the order of abelian groups which admit Paley type partial difference sets has been discovered in [7]. It is proven that if an abelian group admits a Paley type partial difference set and its order is not a prime power, then its order is n^4 or $9n^4$, where $n > 1$ is an odd integer.

Davis [2, Corollary 3.1] constructed a Paley type partial difference set in the abelian group $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ as follows. Let $G = \langle(i, j) | i, j = 0, 1, \ldots, p^2 - 1\rangle$ and C be the set of the elements of order p^2 in the following subgroups of order p^2.

$$\{\langle(1,1)\rangle, \langle(1,2)\rangle, \ldots, \langle(1, \frac{p(p-1)}{2})\rangle, \langle(p,1)\rangle, \langle(2p,1)\rangle, \ldots, \langle(\frac{(p-1)p}{2}, 1)\rangle\},$$

and D be the set of all elements without the identity element in the following subgroups of order p^2.

$$\{\langle(1,0)\rangle, \langle(0,1)\rangle, \langle(1, \frac{p^2-p}{2}+1)\rangle, \ldots, \langle(1, \frac{p^2+1}{2}-2)\rangle\}.$$

Then $S = C \cup D$ is a Paley type partial difference set in the group G which is inverse-closed. We checked it with GAP [9] that the Cayley graph $Cay(G, S)$ is a self-complementary strongly regular graph for $p = 3$. This shows that self-complementary strongly regular Cayley graphs over abelian groups are not on the elementary abelian groups and answers the second part of Question 1.1. But this construction is not self-complementary in general because is not self-complementary for $p = 5$.

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e-mail: M.jazaeri@scu.ac.ir