8th International Conference on
Combinatorics, Cryptography, Computer Science and Computation
November. 15-16, 2023

Direct sum of $AFCMS(G)$

Rasul Rasuli[1]
Department of Mathematics, Payame Noor University(PNU), P. O. Box 19395-4697, Tehran, Iran
Hossien Naraghi
Department of Mathematics, Payame Noor University(PNU), P. O. Box 19395-4697, Tehran, Iran
Bahman Taherkhani
Department of Mathematics, Payame Noor University(PNU), P. O. Box 19395-4697, Tehran, Iran

Abstract

The objective of this article is to present the notion of direct sum of two anti fuzzy multigroups under t-conorms. We show that the direct sum of them is also anti fuzzy multigroup under t-conorms and discuss its various algebraic aspects. We also define strong upper alpha-cut, weak upper alpha-cut, strong lower alpha-cut and weak lower alpha-cut of them and prove some fundamental result of this phenomena. We show that the homomorphic image (pre image) of them will be anti fuzzy multigroups under t-conorms by using the notion of classical homomorphism.

Keywords: Fuzzy multigroups, norms, direct sums, homomorphisms.
AMS Mathematical Subject Classification [2010]: 20N25 , 47A30 , 03E72, 20K30.

1 Introduction

In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object are allowed in a set, then the mathematical structure is called as multiset. Thus, a multiset differs from a set in the sense that each element has a multiplicity. A complete account of the development of multiset theory can be seen in [1, 2, 18, 19]. The concept of fuzzy sets was proposed by Zaded [20] to capture uncertainty in a collection which was neglected in crisp set. Fuzzy set has grown stupendously over the years giving birth to fuzzy groups introduced in [16]. Recently, Shinoj et al. [17] introduced a non-classical group called fuzzy multigroup which generalized fuzzy group. The First author by using norms, investigated some properties of fuzzy algebraic structures [3-15] specially in [3, 4, 5] initiated the study of fuzzy multigroups and anti fuzzy multigroups under norms and investigated some properties of them. We organized this paper as follows: Section 2 contains the introductory definition of multisets, fuzzy multisets, conjugates, commutatives, conorms, anti fuzzy multigroups under t-conorms and related result which plays a key role for our further discussion. In section 3 we define of the direct sum two anti fuzzy multigroups under t-conorms. We prove that the direct sum of them is anti fuzzy multigroups under t-conorms and

[1]speaker

also the fundamental properties of them are discussed deeply in this section. Next we explicate the strong upper alpha-cut, weak upper alpha-cut, strong lower alpha-cut and weak lower alpha-cut of them and also investigate the algebraic properties of this phenomena. Finally, we investigate the notion of them under group homomorphisms.

2 Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel. For details we refer to [3, 4, 5].

Definition 2.1. Let $X = \{x_1, x_2, ..., x_n, ...\}$ be a set. A multiset A over X is a cardinal-valued function, that is, $C_A : X \to \mathbb{N}$ such that $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$, denotes the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$. The set X is called the ground or generic set of the class of all multisets (for short, msets) containing objects from X. A multiset $A = [a, b, b, c, c, c]$ can be represented as $A = [a, b, c]_{1,2,3}$ or $A = [a^1, b^2, c^3]$ or $\{\frac{a}{1}, \frac{b}{2}, \frac{c}{3}\}$. Difierent forms of representing multiset exist other than this. See [10, 20, 30] for details. We denote the set of all multisets by $MS(X)$.

Definition 2.2. Let A and B be two multisets over X, then A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. Note that a multiset is called the parent in relation to its submultiset. Also two multisets A and B over X are comparable to each other if $A \subseteq B$ or $B \subseteq A$.

Definition 2.3. Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0, 1]$. The set of all fuzzy subsets of G is called the $[0, 1]$-power set of G and is denoted $[0, 1]^G$.

Definition 2.4. Let X be a set. A fuzzy multiset A of X is characterized by a count membership function

$$CM_A : X \to [0, 1]$$

of which the value is a multiset of the unit interval $I = [0, 1]$. That is,

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n, ...\} \forall x \in X,$$

where $\mu^1, \mu^2, ..., \mu^n, ... \in [0, 1]$ such that

$$(\mu^1 \geq \mu^2 \geq ... \geq \mu^n \geq ...).$$

Whenever the fuzzy multiset is finite, we write

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n\},$$

where $\mu^1, \mu^2, ..., \mu^n \in [0, 1]$ such that

$$(\mu^1 \geq \mu^2 \geq ... \geq \mu^n),$$

or simply

$$CM_A(x) = \{\mu^i\},$$

for $\mu^i \in [0,1]$ and $i = 1, 2, ..., n$.
Now, a fuzzy multiset A is given as
$A = \{\frac{CM_A(x)}{x} : x \in X\}$ or $A = \{(CM_A(x), x) : x \in X\}$.
The set of all fuzzy multisets is depicted by $FMS(X)$.

Example 2.5. Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{1, 0.5, 0.4\}$ and $CM_A(b) = \{0.9, 0.6\}$ and $CM_A(c) = \{0\}$ we get that A is a fuzzy multiset of X written as

$$A = \{\frac{1, 0.5, 0.4}{a}, \frac{0.9, 0.6}{b}\}.$$

Definition 2.6. Let $A, B \in FMS(X)$. Then A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.7. Let $A \in FMS(X)$ and $\alpha \in [0,1]$. Then we define
(1) $A^\star = \{x \in X \mid CM_A(x) = CM_A(e_X)\}$ where e_X is the identity element of X.
(2) $A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha\}$ is called strong upper alpha-cut of A.
(3) $A_{(\alpha)} = \{x \in X \mid CM_A(x) > \alpha\}$ is called weak upper alpha-cut of A.
(4) $A^{[\alpha]} = \{x \in X \mid CM_A(x) \leq \alpha\}$ is called strong lower alpha-cut of A.
(6) $A^{(\alpha)} = \{x \in X \mid CM_A(x) < \alpha\}$ is called weak lower alpha-cut of A.

Definition 2.8. Let $A, B \in FMS(G)$. We say A is conjugate to B if for all $x, y \in G$ we have that $CM_A(x) = CM_B(yxy^{-1})$.

Definition 2.9. Let $A \in FMS(G)$. We say A is commutative if $CM_A(xy) = CM_A(yx)$ for all $x, y \in G$.

Definition 2.10. A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:
(C1) $C(x, 0) = x$
(C2) $C(x, y) \leq C(x, z)$ if $y \leq z$
(C3) $C(x, y) = C(y, x)$
(C4) $C(x, C(y, z)) = C(C(x, y), z)$,
for all $x, y, z \in [0,1]$.

Example 2.11. (1) Standard union t-conorm $C_m(x, y) = \max\{x, y\}$.
(2) Bounded sum t-conorm $C_b(x, y) = \min\{1, x + y\}$.
(3) Algebraic sum t-conorm $C_p(x, y) = x + y - xy$.
(4) Drastic T-conorm

$$C_D(x, y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic T-norm.
(5) Nilpotent maximum T-conorm , dual to the nilpotent minimum T-norm:

$$C_{nM}(x, y) = \begin{cases} \max\{x, y\} & \text{if } x + y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x, y) = \frac{x + y}{1 + xy}$ is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x, y) \leq C(x, y) \leq C_D(x, y)$ for any t-conorm C and all $x, y \in [0,1]$.

Recall that t-conorm C is idempotent if for all $x \in [0,1]$, we have that $C(x,x) = x$.

Lemma 2.12. *Let C be a t-conorm. Then*

$$C(C(x,y), C(w,z)) = C(C(x,w), C(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Definition 2.13. Let $A \in FMS(G)$. We say A is an anti fuzzy multigroup of G under t-conorm C if it satisfies the following two conditions:
(1) $CM_A(xy) \leq C(CM_A(x), CM_A(y))$,
(2) $CM_A(x^{-1}) \leq CM_A(x)$,
for all $x, y \in G$.
The set of all anti fuzzy multisets of G under t-conorm C is depicted by $AFCMS(G)$.

Theorem 2.14. *Let $A \in AFCMS(G)$. If C be idempotent t-conorm, then for all $x \in G$, and $n \geq 1$,*
(1) $CM_A(e) \leq CM_A(x)$;
(2) $CM_A(x^n) \leq CM_A(x)$;
(3) $CM_A(x) = CM_A(x^{-1})$.

3 Direct sum of $AFCMS(G)$

Definition 3.1. Let $A \in AFCMS(G)$ and $B \in AFCMS(H)$. The direct sum of A and B, denoted by $A \oplus B$, is characterized by a count membership function

$$CM_{A\oplus B} : G \oplus H \to [0,1]$$

such that

$$CM_{A\oplus B}(x,y) = C(CM_A(x), CM_B(y))$$

for all $x \in G$ and $y \in H$.

Example 3.2. Let $G = \{1, x\}$ be a group, where $x^2 = 1$ and $H = \{e, a, b, c\}$ be a Klein 4-group, where $a^2 = b^2 = c^2 = e$. Suppose

$$A = \{\frac{0.6, 0.4, 0.2}{1}, \frac{1, 0.1}{x}\}$$

and

$$B = \{\frac{0.9, 0.35}{e}, \frac{0.55, 0.45, 0.25}{a}, \frac{0.80, 0.55}{b}, \frac{0.6, 0.3}{c}\}$$

be fuzzy multigroups of G and H. Let

$$G \oplus H = \{(1,e), (1,a), (1,b), (1,c), (x,e), (x,a), (x,b), (x,c)\}$$

be a a group from the classical sense. Define
$A \oplus B = \{\frac{1, 0.55, 0.25}{(1,e)}, \frac{1, 0.45}{(1,a)}, \frac{0.30, 0.2}{(1,b)}, \frac{0.55, 0.35, 0.15}{(1,c)}, \frac{0.75, 0.65}{(x,e)}, \frac{1, 0.25, 0.15}{(x,a)}, \frac{0.10, 0.50}{(x,b)}, \frac{0.7, 0.6}{(x,c)}\}$ and let $C(x,y) = C_p(x,y) = x + y - xy$ for all $x, y \in [0,1]$. Then

$$A \oplus B \in AFCMS(G \oplus H).$$

Proposition 3.3. *Let $A_i \in AFCMS(G_i)$ for $i = 1, 2$. Then $A_1 \oplus A_2 \in AFCMS(G_1 \oplus G_2)$.*

Proof. Let $(a_1, b_1), (a_2, b_2) \in G_1 \oplus G_2$. Then

$$\begin{aligned}
(CM_{A\oplus B})((a_1, b_1)(a_2, b_2)) &= (CM_{A\oplus B})(a_1a_2, b_1b_2) \\
&= C(CM_A(a_1a_2), CM_B(b_1b_2)) \\
&\leq C(C(CM_A(a_1), CM_A(a_2)), C(CM_B(b_1), CM_B(b_2))) \\
&= C(C(CM_A(a_1), CM_B(b_1), C(CM_A(a_2), CM_B(b_2)) \ (\textit{Lemma 2.12}) \\
&= C((CM_{A\oplus B})(a_1, b_1), (CM_{A\oplus B})(a_2, b_2))
\end{aligned}$$

then

$$(CM_{A\oplus B})((a_1, b_1)(a_2, b_2)) \leq C((CM_{A\oplus B})(a_1, b_1), (CM_{A\oplus B})(a_2, b_2)).$$

Also

$$\begin{aligned}
(CM_{A\oplus B})(a_1, b_1)^{-1} &= (CM_{A\oplus B})(a_1^{-1}, b_1^{-1}) \\
&= C(CM_A(a_1^{-1}), CM_B(b_1^{-1})) \\
&\leq C(CM_A(a_1), CM_B(b_1)) \\
&= (CM_{A\oplus B})(a_1, b_1)
\end{aligned}$$

thus

$$(CM_{A\oplus B})(a_1, b_1)^{-1} \leq (CM_{A\oplus B})(a_1, b_1).$$

Then $A_1 \oplus A_2 \in AFCMS(G_1 \oplus G_2)$. □

Corollary 3.4. *Let $A \in AFCMS(G)$ and $B \in AFCMS(H)$. Then*

$$A \oplus 1_H, 1_G \oplus B \in AFCMS(G \oplus H).$$

Corollary 3.5. *Let $A_i \in AFCMS(G_i)$ for $i = 1, 2, ..., n$. Then*

$$A_1 \oplus A_2 \oplus ... \oplus A_n \in AFCMS(G_1 \oplus G_2 \oplus ... \oplus G_n).$$

Proposition 3.6. *Let $A \in AFCMS(G)$ and $B \in AFCMS(H)$ such that C be idempotent t-conorm. Then for all $\alpha \in [0, 1]$ the following assertions hold.*
(1) $(A \oplus B)^\star = A^\star \oplus B^\star$.
(2) $(A \oplus B)_{[\alpha]} = A_{[\alpha]} \oplus B_{[\alpha]}$.
(3) $(A \oplus B)_{(\alpha)} = A_{(\alpha)} \oplus B_{(\alpha)}$.
(4) $(A \oplus B)^{[\alpha]} = A^{[\alpha]} \oplus B^{[\alpha]}$.
(5) $(A \oplus B)^{(\alpha)} = A^{(\alpha)} \oplus B^{(\alpha)}$.

Proof. (1) As $(A \oplus B)^\star = \{(x, y) \in G \oplus H \mid CM_{A\oplus B}(x, y) = CM_{A\oplus B}(e_G, e_H)\}$ so

$$(x, y) \in (A \oplus B)^\star \Longleftrightarrow CM_{A\oplus B}(x, y) = CM_{A\oplus B}(e_G, e_H)$$

$$\Longleftrightarrow C(CM_A(x), CM_B(y)) = C(CM_A(e_G), CM_B(e_H))$$

$$\Longleftrightarrow CM_A(x) = CM_A(e_G)\ and\, CM_B(y) = CM_B(e_H)$$

$$\Longleftrightarrow x \in A^\star\ and\, y \in B^\star \Longleftrightarrow (x, y) \in A^\star \oplus B^\star$$

thus

$$(A \oplus B)^\star = A^\star \oplus B^\star.$$

(2) Let $(A \oplus B)_{[\alpha]} = \{(x, y) \in G \oplus H \mid CM_{A\oplus B}(x, y) \geq \alpha\}$. Now

$$(x, y) \in (A \oplus B)_{[\alpha]} \Longleftrightarrow CM_{A\oplus B}(x, y) \geq \alpha \Longleftrightarrow C(CM_A(x), CM_B(y)) \geq \alpha$$

$$\Longleftrightarrow C(CM_A(x), CM_B(y)) \geq \alpha = C(\alpha, \alpha) \Longleftrightarrow CM_A(x) \geq \alpha\ and\, CM_B(y) \geq \alpha$$

$$\Longleftrightarrow x \in A_{[\alpha]}\ and\, y \in B_{[\alpha]} \Longleftrightarrow (x, y) \in A_{[\alpha]} \oplus B_{[\alpha]}$$

thus

$$(A \oplus B)_{[\alpha]} = A_{[\alpha]} \oplus B_{[\alpha]}.$$

(3) Since $(A \oplus B)_{(\alpha)} = \{(x, y) \in G \oplus H \mid CM_{A\oplus B}(x, y) > \alpha\}$ so

$$(x, y) \in (A \oplus B)_{(\alpha)} \Longleftrightarrow CM_{A\oplus B}(x, y) > \alpha \Longleftrightarrow C(CM_A(x), CM_B(y)) > \alpha$$

$$\Longleftrightarrow C(CM_A(x), CM_B(y)) > \alpha = C(\alpha, \alpha) \Longleftrightarrow CM_A(x) > \alpha\ and\, CM_B(y) > \alpha$$

$$\Longleftrightarrow x \in A_{(\alpha)}\ and\, y \in B_{(\alpha)} \Longleftrightarrow (x, y) \in A_{(\alpha)} \oplus B_{(\alpha)}$$

and so

$$(A \oplus B)_{(\alpha)} = A_{(\alpha)} \oplus B_{(\alpha)}.$$

(4) Because $(A \oplus B)^{[\alpha]} = \{(x, y) \in G \oplus H \mid CM_{A\oplus B}(x, y) \leq \alpha\}$ then

$$(x, y) \in (A \oplus B)^{[\alpha]} \Longleftrightarrow CM_{A\oplus B}(x, y) \leq \alpha \Longleftrightarrow C(CM_A(x), CM_B(y)) \leq \alpha$$

$$\Longleftrightarrow C(CM_A(x), CM_B(y)) \leq \alpha = C(\alpha, \alpha) \Longleftrightarrow CM_A(x) \leq \alpha\ and\, CM_B(y) \leq \alpha$$

$$\Longleftrightarrow x \in A^{[\alpha]}\ and\, y \in B^{[\alpha]} \Longleftrightarrow (x, y) \in A^{[\alpha]} \oplus B^{[\alpha]}$$

therefore

$$(A \oplus B)^{[\alpha]} = A^{[\alpha]} \oplus B^{[\alpha]}.$$

(5) Because of $(A \oplus B)^{(\alpha)} = \{(x, y) \in G \oplus H \mid CM_{A\oplus B}(x, y) < \alpha\}$ then

$$(x, y) \in (A \oplus B)^{(\alpha)} \Longleftrightarrow CM_{A\oplus B}(x, y) < \alpha \Longleftrightarrow C(CM_A(x), CM_B(y)) < \alpha$$

$$\Longleftrightarrow C(CM_A(x), CM_B(y)) < \alpha = C(\alpha, \alpha) \Longleftrightarrow CM_A(x) < \alpha\ and\, CM_B(y) < \alpha$$

$$\Longleftrightarrow x \in A^{(\alpha)}\ and\, y \in B^{(\alpha)} \Longleftrightarrow (x, y) \in A^{(\alpha)} \oplus B^{(\alpha)}$$

then

$$(A \oplus B)^{(\alpha)} = A^{(\alpha)} \oplus B^{(\alpha)}.$$

□

Proposition 3.7. *Let $A \in AFCMS(G)$ and $B \in AFCMS(H)$ such that C be idempotent t-conorm. Then for all $(x, y) \in G \times H$ the following assertions hold.*
(1) $CM_{A\oplus B}(e_G, e_H) \leq CM_{A\oplus B}(x, y)$.
(2) $CM_{A\oplus B}((x, y)^n) \leq CM_{A\oplus B}(x, y)$.
(3) $CM_{A\oplus B}(x, y) = CM_{A\oplus B}(x^{-1}, y^{-1})$.

Proof. Using Proposition 3.3 we get that $A \oplus B \in AFCMS(G \oplus H)$. Now Theorem 2.14 gives us that assertions are hold. □

Proposition 3.8. *Let $A \in AFCMS(G)$ and $B \in AFCMS(H)$ such that C be idempotent t-conorm. Then for all $\alpha \in [0, 1]$ the following assertions hold.*
(1) $(A \oplus B)^\star$ is a subgroup of $G \oplus H$.
(2) $(A \oplus B)^{[\alpha]}$ is a subgroup of $G \oplus H$.
(3) $(A \oplus B)^{(\alpha)}$ is a subgroup of $G \oplus H$.

Proof. (1) Let $(x_1, y_1), (x_2, y_2) \in (A \oplus B)_\star$ and we must prove that $(x_1, y_1)(x_2, y_2)^{-1} \in (A \oplus B)^\star$. Because $(x_1, y_1), (x_2, y_2) \in (A \oplus B)^\star$ then

$$CM_{A\oplus B}(x_1, y_1) = CM_{A\oplus B}(x_2, y_2) = CM_{A\oplus B}(e_G, e_H)$$

which means that

$$C(CM_A(x_1), CM_B(y_1)) = C(CM_A(x_2), CM_B(y_2)) = C(CM_A(e_G), CM_B(e_H))$$

and so $CM_A(x_1) = CM_A(x_2) = CM_A(e_G)$ and $CM_A(y_1) = CM_A(y_2) = CM_A(e_H)$. Then

$$\begin{aligned}
CM_{A\oplus B}((x_1, y_1)(x_2, y_2)^{-1}) &= CM_{A\oplus B}((x_1, y_1)(x_2^{-1}, y_2^{-1})) \\
&= CM_{A\oplus B}(x_1x_2^{-1}, y_1y_2^{-1}) \\
&= C(CM_A(x_1x_2^{-1}), CM_B(y_1y_2^{-1})) \\
&\leq C(C(CM_A(x_1), CM_A(x_2^{-1})), C(CM_B(y_1), CM_B(y_2^{-1}))) \\
&\leq C(C(CM_A(x_1), CM_A(x_2)), C(CM_B(y_1), CM_B(y_2))) \\
&= C(C(CM_A(e_G), CM_A(e_G)), C(CM_B(e_H), CM_B(e_H))) \\
&= C(CM_A(e_G), CM_B(e_H)) = CM_{A\oplus B}(e_G, e_H) \\
&\leq CM_{A\oplus B}((x_1, y_1)(x_2, y_2)^{-1}) \text{ (Proposition 3.7 paret(1))}
\end{aligned}$$

thus $CM_{A\oplus B}((x_1, y_1)(x_2, y_2)^{-1}) = CM_{A\oplus B}(e_G, e_H)$ and so $(x_1, y_1)(x_2, y_2)^{-1} \in (A \oplus B)^\star$. Now we obtain that $(A \oplus B)^\star$ is a subgroup of $G \oplus H$.
(2) Let $(x_1, y_1), (x_2, y_2) \in (A \oplus B)^{[\alpha]}$ and we show that $(x_1, y_1)(x_2, y_2)^{-1} \in (A \oplus B)^{[\alpha]}$. As $(x_1, y_1), (x_2, y_2) \in (A \oplus B)^{[\alpha]}$ so $CM_{A\oplus B}(x_1, y_1) \leq \alpha$ and $CM_{A\oplus B}(x_2, y_2) \leq \alpha$. Now

$$\begin{aligned}
CM_{A\oplus B}((x_1,y_1)(x_2,y_2)^{-1}) &= CM_{A\oplus B}((x_1,y_1)(x_2^{-1},y_2^{-1}))\\
&= CM_{A\oplus B}(x_1x_2^{-1},y_1y_2^{-1})\\
&= C(CM_A(x_1x_2^{-1}),CM_B(y_1y_2^{-1}))\\
&\leq C(C(CM_A(x_1),CM_A(x_2^{-1})),C(CM_B(y_1),CM_B(y_2^{-1})))\\
&\leq C(C(CM_A(x_1),CM_A(x_2)),C(CM_B(y_1),CM_B(y_2)))\\
&= C(C(CM_A(x_1),CM_B(y_1)),C(CM_A(x_2),CM_B(y_2)))\ (\textit{Lemma 2.12})\\
&= C(CM_{A\oplus B}(x_1,y_1),CM_{A\oplus B}(x_2,y_2))\\
&\leq C(\alpha,\alpha)=\alpha
\end{aligned}$$

and thus $CM_{A\oplus B}((x_1,y_1)(x_2,y_2)^{-1}) \leq \alpha$ which means that $(x_1,y_1)(x_2,y_2)^{-1} \in (A\oplus B)^{[\alpha]}$. Then $(A\oplus B)^{[\alpha]}$ is a subgroup of $G\oplus H$.
(3) If $(x_1,y_1),(x_2,y_2) \in (A\oplus B)^{(\alpha)}$, then $CM_{A\oplus B}(x_1,y_1) < \alpha$ and $CM_{A\oplus B}(x_2,y_2) < \alpha$. Then

$$\begin{aligned}
CM_{A\oplus B}((x_1,y_1)(x_2,y_2)^{-1}) &= CM_{A\oplus B}((x_1,y_1)(x_2^{-1},y_2^{-1}))\\
&= CM_{A\oplus B}(x_1x_2^{-1},y_1y_2^{-1})\\
&= C(CM_A(x_1x_2^{-1}),CM_B(y_1y_2^{-1}))\\
&\leq C(C(CM_A(x_1),CM_A(x_2^{-1})),T(CM_B(y_1),CM_B(y_2^{-1})))\\
&\leq C(C(CM_A(x_1),CM_A(x_2)),C(CM_B(y_1),CM_B(y_2)))\\
&= C(C(CM_A(x_1),CM_B(y_1)),C(CM_A(x_2),CM_B(y_2)))\ (\textit{Lemma 2.12})\\
&= C(CM_{A\oplus B}(x_1,y_1),CM_{A\oplus B}(x_2,y_2))\\
&< C(\alpha,\alpha)=\alpha
\end{aligned}$$

and thus $CM_{A\oplus B}((x_1,y_1)(x_2,y_2)^{-1}) < \alpha$ which means that $(x_1,y_1)(x_2,y_2)^{-1} \in (A\oplus B)^{(\alpha)}$. Then $(A\oplus B)^{(\alpha)}$ is a subgroup of $G\oplus H$. □

Proposition 3.9. *Let $A \in AFCMS(G)$ and $B \in AFCMS(H)$. If $A\oplus B \in AFCMS(G\oplus H)$, then at least one of the following statements hold.*
(1) $CM_B(e_H)) \leq CM_A(x)$ for all $x \in G$.
(2) $CM_A(e_G)) \leq CM_B(y)$ for all $y \in H$.

Proof. By contrapositive, suppose that none of the statements holds. Then suppose we can find $a \in G$ and $b \in H$ such that $CM_A(a) < CM_B(e_H)$ and $CM_B(b) < CM_A(e_G)$. Now

$$\begin{aligned}
CM_{A\oplus B}(a,b) &= C(CM_A(a),CM_B(b))\\
&< T(CM_B(e_H),CM_A(e_G))\\
&= C(CM_A(e_G),CM_B(e_H))\\
&= CM_{A\oplus B}(e_G,e_H)
\end{aligned}$$

and thus $CM_{A\oplus B}(a,b) < CM_{A\oplus B}(e_G,e_H)$ and this is contradiction with Proposition 3.7 part (1). Then at least one of the statements hold. □

Proposition 3.10. *Let $A \in FMS(G)$ and $B \in FMS(H)$. Let $A \oplus B \in AFCMS(G \oplus H)$ and $CM_A(x) \geq CM_B(e_H)$ for all $x \in G$. Then $A \in AFCMS(G)$.*

Proof. As $CM_A(x) \geq CM_B(e_H)$ for all $x \in G$ so $CM_A(y) \geq CM_B(e_H)$ and $CM_A(xy) \geq CM_B(e_H) = CM_B(e_H e_H)$ for all $y \in G$. Then

$$\begin{aligned} CM_A(xy) &= C(CM_A(xy), CM_B(e_H e_H)) \\ &= CM_{A\oplus B}(xy, e_H e_H) \\ &= CM_{A\oplus B}((x, e_H)(y, e_H)) \\ &\leq C(CM_{A\oplus B}(x, e_H), CM_{A\oplus B}(y, e_H)) \\ &= C(C(CM_A(x), CM_B(e_H)), C(CM_A(y), CM_B(e_H))) \\ &= C(CM_A(x), CM_A(y)) \end{aligned}$$

and so

$$CM_A(xy) \leq C(CM_A(x), CM_A(y)).$$

Also since $CM_A(x) \geq CM_B(e_H)$ for all $x \in G$ so $CM_A(x^{-1}) \geq CM_B(e_H)$. Thus

$$\begin{aligned} CM_A(x^{-1}) &= C(CM_A(x^{-1}), CM_A(e_H)) \\ &= C(CM_A(x^{-1}), CM_A(e_H^{-1})) \\ &= CM_{A\oplus B}((x, e_H)^{-1}) \\ &\leq CM_{A\oplus B}(x, e_H) \\ &= C(CM_A(x), CM_A(e_H)) \\ &= CM_A(x) \end{aligned}$$

and then $CM_A(x^{-1}) \leq CM_A(x)$. Therefore $A \in AFCMS(G)$. □

Proposition 3.11. *Let $A \in FMS(G)$ and $B \in FMS(H)$. Let $A \oplus B \in AFCMS(G \oplus H)$ and $CM_B(x) \geq CM_A(e_G)$ for all $x \in H$. Then $B \in AFCMS(H)$.*

Proof. The proof is similar to Proposition 3.10. □

Corollary 3.12. *Let $A \in FMS(G)$ and $B \in FMS(H)$ such that $A \oplus B \in AFCMS(G \oplus H)$. Then either $A \in AFCMS(G)$ or $B \in AFCMS(H)$.*

Proof. Using Proposition 3.9 we get that $CM_B(e_H)) \leq CM_A(x)$ for all $x \in G$ or $CM_A(e_G)) \leq CM_B(y)$ for all $y \in H$. Then from Proposition 3.10 and Proposition 3.11 we will have that either $A \in AFCMS(G)$ or $B \in AFCMS(H)$. □

Proposition 3.13. *Let $A, C \in AFCMS(G)$ and $B, D \in AFCMS(H)$. If A is conjugate to B and C is conjugate to D, then $A \oplus C$ is conjugate to $B \oplus D$.*

Proof. As A is conjugate to B so $CM_A(x) = CM_C(gxg^{-1})$ and as B is conjugate to D so $CM_B(y) = CM_D(hyh^{-1})$ for all $x, g \in G$ and $y, h \in H$. Now

$$\begin{aligned} CM_{A\oplus B}(x,y) &= C(CM_A(x), CM_B(y)) \\ &= C(CM_C(gxg^{-1}), CM_D(hyh^{-1})) \\ &= CM_{C\oplus D}(gxg^{-1}, hyh^{-1}) \\ &= CM_{C\oplus D}((g,h)(x,y)(g^{-1},h^{-1})) \\ &= CM_{C\oplus D}((g,h)(x,y)(g,h)^{-1}) \end{aligned}$$

and thus $CM_{A\oplus B}(x,y) = CM_{C\oplus D}((g,h)(x,y)(g,h)^{-1})$ which means that $A\oplus C$ is conjugate to $B\oplus D$. □

Proposition 3.14. *Let $A \in AFCMS(G)$ and $B \in AFCMS(H)$. Then A and B are commutatives if and only if $A \oplus B$ is a commutative.*

Proof. Let $x_1, y_1 \in G$ and $x_2, y_2 \in H$ such that $x = (x_1, x_2) \in G \times H$ and $y = (y_1, y_2) \in G \times H$. Let A and B are commutative then $CM_A(x_1y_1) = CM_A(y_1x_1)$ and $CM_B(x_2y_2) = CM_B(y_2x_2)$. Then

$$\begin{aligned} CM_{A\oplus B}(xy) &= CM_{A\oplus B}((x_1,x_2)(y_1,y_2)) \\ &= CM_{A\oplus B}(x_1y_1, x_2y_2) \\ &= C(CM_A(x_1y_1), CM_B(x_2y_2)) \\ &= C(CM_A(y_1x_1), CM_B(y_2x_2)) \\ &= CM_{A\oplus B}(y_1x_1, y_2x_2) \\ &= CM_{A\oplus B}((y_1,y_2)(x_1,x_2)) \\ &= CM_{A\oplus B}(yx) \end{aligned}$$

thus $CM_{A\oplus B}(xy) = CM_{A\oplus B}(yx)$ and then $A \oplus B$ is a commutative.
Conversely, suppose that $A \oplus B$ is a commutative. Then

$$CM_{A\oplus B}(xy) = CM_{A\oplus B}(yx)$$

$$\iff CM_{A\oplus B}((x_1,x_2)(y_1,y_2)) = CM_{A\oplus B}((y_1,y_2)(x_1,x_2))$$

$$\iff CM_{A\oplus B}(x_1y_1, x_2y_2) = CM_{A\oplus B}(y_1x_1, y_2x_2)$$

$$\iff C(CM_A(x_1y_1), CM_B(x_2y_2)) = C(CM_A(y_1x_1), CM_B(y_2x_2))$$

$$\iff CM_A(x_1y_1) = CM_A(y_1x_1) \textit{ and } CM_B(x_2y_2) = CM_B(y_2x_2)$$

which gives us that A and B are commutative. □

Definition 3.15. Let $G\oplus H$ and $I \oplus J$ be groups and $f : G\oplus H \to I \oplus J$ be a homomorphism. Let $A\oplus B \in FMS(G\oplus H)$ and $C \oplus D \in FMS(I \oplus J)$. Define $f(A\oplus B) \in FMS(I \oplus J)$ and $f^{-1}(C \oplus D) \in FMS(G\oplus H)$ as

$$f(CM_{A\oplus B})(i,j) = (CM_{f(A\oplus B)})(i,j)$$
$$= \begin{cases} \inf\{CM_{A\oplus B}(g,h) \mid g \in G, h \in H, f(g,h) = (i,j)\} & \text{if } f^{-1}(i,j) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^{-1}(CM_{C\oplus D}(g,h)) = CM_{f^{-1}(C\oplus D)}(g,h) = CM_{C\oplus D}(f(g,h))$$

for all $(g,h) \in G \oplus H$.

Proposition 3.16. *Let $G \oplus H$ and $I \oplus J$ be groups and $f : G \oplus H \to I \oplus J$ be an epimorphism. If $A \in AFCMS(G), B \in AFCMS(H)$ and $A \oplus B \in AFCMS(G \oplus H)$, then $f(A \oplus B) \in AFCMS(I \oplus J)$.*

Proof. (1) Let $X = (i_1, j_1) \in I \oplus J$ and $Y = (i_2, j_2) \in I \oplus J$ such that

$$f^{-1}(XY) = f^{-1}((i_1,j_1)(i_2,j_2)) = f^{-1}(i_1i_2, j_1j_2) \neq \emptyset.$$

Then

$f(A \oplus B)(XY) = f(A \oplus B)((i_1,j_1)(i_2,j_2)) = f(A \oplus B)(i_1i_2, j_1j_2)$
$= \inf\{CM_{A\oplus B}(g_1g_2, h_1h_2) : g_1, g_2 \in G, h_1, h_2 \in H, f(g_1g_2, h_1h_2) = (i_1i_2, j_1j_2)\}$
$= \inf\{CM_{A\oplus B}(g_1g_2, h_1h_2) : g_1, g_2 \in G, h_1, h_2 \in H, (f(g_1g_2), f(h_1h_2)) = (i_1i_2, j_1j_2)\}$
$= \inf\{CM_{A\oplus B}(g_1g_2, h_1h_2) : g_1, g_2 \in G, h_1, h_2 \in H, f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\}$
$= \inf\{C(CM_A(g_1g_2), CM_B(h_1h_2)) : g_1, g_2 \in G, h_1, h_2 \in H, f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\}$
$\leq \inf\{C(C(CM_A(g_1), CM_A(g_2)), T(CM_B(h_1), CM_B(h_2))) : f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\}$
$= \inf\{C(C(CM_A(g_1), CM_B(h_1)), C(CM_A(g_2), CM_B(h_2))) : f(g_1g_2) = i_1i_2, f(h_1h_2) = j_1j_2\}$
$= \inf\{C(C(CM_A(g_1), CM_B(h_1)), C(CM_A(g_2), CM_B(h_2))) : f(g_1) = i_1, f(g_2) = i_2, f(h_1) = j_1, f(h_2) = j_2\}$
$= \inf\{C(CM_{A\oplus B}(g_1, h_1), CM_{A\times B}(g_2, h_2)) : f(g_1) = i_1, f(g_2) = i_2, f(h_1) = j_1, f(h_2) = j_2\}$
$= C(\inf\{CM_{A\times B}(g_1, h_1) : f(g_1, h_1) = (i_1, j_1)\}, \inf\{CM_{A\times B}(g_2, h_2) \mid f(g_2, h_2) = (i_2, j_2)\})$
$= C(f(A \oplus B)(i_1, j_1), f(A \times B)(i_2, j_2))$
$= C(f(A \oplus B)(X), f(A \oplus B)(Y))$

thus

$$f(A \oplus B)(XY) \leq C(f(A \oplus B)(X), f(A \oplus B)(Y)).$$

(2) Let $X = (i,j) \in I \oplus J$ then

$$\begin{aligned}
f(A \oplus B)(X^{-1}) &= f(A \oplus B)((i,j)^{-1}) = f(A \oplus B)(i^{-1}, j^{-1}) \\
&= \inf\{CM_{A\oplus B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, f(g^{-1}, h^{-1}) = (i^{-1}, j^{-1})\} \\
&= \inf\{CM_{A\oplus B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, (f(g^{-1}), f(h^{-1})) = (i^{-1}, j^{-1})\} \\
&= \inf\{CM_{A\oplus B}(g^{-1}, h^{-1}) \mid g \in G, h \in H, f(g^{-1}) = i^{-1}, f(h^{-1})) = j^{-1}\} \\
&= \inf\{C(CM_A(g^{-1}), CM_B(h^{-1})) \mid g \in G, h \in H, f(g^{-1}) = i^{-1}, f(h^{-1})) = j^{-1}\} \\
&\leq \inf\{C(CM_A(g), CM_B(h)) \mid g \in G, h \in H, f^{-1}(g) = i^{-1}, f^{-1}(h) = j^{-1}\} \\
&= \inf\{C(CM_A(g), CM_B(h)) \mid g \in G, h \in H, f(g) = i, f(h) = j\} \\
&= \inf\{CM_{A\oplus B}(g,h) \mid (g,h) \in G \oplus H, f(g,h) = (i,j)\} \\
&= f(A \oplus B)(i,j) = f(A \oplus B)(X)
\end{aligned}$$

and then

$$f(A \oplus B)(X^{-1}) \leq f(A \oplus B)(X).$$

Therefore $f(A \oplus B) \in AFCMS(I \oplus J)$. □

Proposition 3.17. *Let $G \oplus H$ and $I \oplus J$ be groups and $f : G \oplus H \to I \oplus J$ be a homomorphism. If $C \in AFCMS(I)$ and $D \in AFCMS(J)$ and $C \oplus D \in AFCMS(I \oplus J)$, then $f^{-1}(C \oplus D) \in AFCMS(G \oplus H)$.*

Proof. (1) Let $X = (g_1, h_1) \in G \oplus H$ and $Y = (g_2, h_2) \in G \oplus H$. Then

$$\begin{aligned}
f^{-1}(CM_{C\oplus D})(XY) &= f^{-1}(CM_{C\oplus D})((g_1,h_1)(g_2,h_2)) \\
&= f^{-1}(CM_{C\oplus D})(g_1g_2, h_1h_2)) \\
&= CM_{C\oplus D}(f(g_1g_2, h_1h_2)) \\
&= CM_{C\oplus D}(f(g_1g_2), f(h_1h_2)) \\
&= C(CM_C(f(g_1g_2)), CM_D(f(h_1h_2))) \\
&= C(CM_C(f(g_1)f(g_2)), CM_D(f(h_1)f(h_2))) \\
&\leq C(C(CM_C(f(g_1)), CM_C(f(g_2))), C(CM_D(f(h_1)), CM_D(f(h_2))) \\
&= C(C(CM_C(f(g_1)), CM_D(f(h_1))), C(CM_C(f(g_2), CM_D(f(h_2))) \\
&= C(CM_{C\oplus D}(f(g_1), f(h_1)), CM_{C\oplus D}(f(g_2), f(h_2))) \\
&= C(CM_{C\oplus D}(f(g_1, h_1)), CM_{C\oplus D}(f(g_2, h_2))) \\
&= C(f^{-1}(CM_{C\oplus D})(g_1, h_1), f^{-1}(CM_{C\oplus D})(g_2, h_2)) \\
&= C(f^{-1}(CM_{C\oplus D})(X), f^{-1}(CM_{C\oplus D})(Y))
\end{aligned}$$

and then

$$f^{-1}(CM_{C\oplus D})(XY) \leq C(f^{-1}(CM_{C\oplus D})(X), f^{-1}(CM_{C\oplus D})(Y)).$$

(2) Let $X = (g, h) \in G \oplus H$. Then

$$\begin{aligned}
f^{-1}(CM_{C\oplus D})(X^{-1}) &= f^{-1}(CM_{C\oplus D})((g_1, h_1)^{-1}) \\
&= CM_{C\oplus D}(f(g,h)^{-1}) \\
&= CM_{C\oplus D}(f(g^{-1}, h^{-1})) \\
&= CM_{C\oplus D}(f^{-1}(g), f^{-1}(h)) \\
&= C(CM_C(f^{-1}(g)), CM_D(f^{-1}(h))) \\
&\leq C(CM_C(f(g)), CM_D(f(h))) \\
&= CM_{C\oplus D}(f(g), f(h)) \\
&= CM_{C\oplus D}(f(g,h)) \\
&= f^{-1}(CM_{C\oplus D})(g,h) \\
&= f^{-1}(CM_{C\oplus D})(X)
\end{aligned}$$

and then

$$f^{-1}(CM_{C\oplus D})(X^{-1}) \leq f^{-1}(CM_{C\oplus D})(X).$$

Thus $f^{-1}(C \oplus D) \in AFCMS(G \oplus H)$. □

4 Open problem

In this study, we introduced the notion of direct sum of two anti fuzzy multigroups under t-conorms and we defined strong upper alpha-cut, weak upper alpha-cut, strong lower alpha-cut and weak lower alpha-cut of them and prove some fundamental result of this phenomena. Now one can study and obtain anti fuzzy multirings under t-conorms as we did for groups and this can be an open problem.

Acknowledgment. It is our pleasant duty to thank referees for their useful suggestions which helped us to improve our manuscript.

References

[1] W. D. Blizard, *Dedekind multisets and function shells*, Theoret. Comput. Sci., **110**(1993), 79-98.

[2] W. D. Blizard, *Multiset theory*, Notre Dame J. Form. Log., **30**(1)(1989), 36-66.

[3] R. Rasuli, *t-norms over Fuzzy Multigroups*, Earthline Journal of Mathematical Sciences, **3**(2)(2020), 207-228.

[4] R. Rasuli, *Direct product of fuzzy multigroups under t-norms*, Open Journal of Discrete Applied Mathematics (ODAM), **3**(1)(2020), 75-85.

[5] R. Rasuli, *Anti fuzzy multigroups and t-conorms*, 2^{th} National Conference on Soft Computing and Cognitive Science held by Conbad Kavous University, Factualy of Technology and Engineering Minutest, Iran, Minudasht, during April 18-19, 2023

[6] R. Rasuli, *Norms Over Intuitionistic Fuzzy Subgroups on Direct Product of Groups*, Commun. Combin., Cryptogr. & Computer Sci., **1**(2023), 39-54.

[7] R. Rasuli, *T-norms over complex fuzzy subgroups*, Mathematical Analysis and its Contemporary Applications, **5(1)**(2023), 33-49.

[8] R. Rasuli, *T-Fuzzy subalgebras of BCI-algebras*, Int. J. Open Problems Compt. Math., **16(1)**(2023), 55-72.

[9] R. Rasuli, *Norms over Q-intuitionistic fuzzy subgroups of a group*, Notes on Intuitionistic Fuzzy Sets, **29(1)**(2023), 30-45.

[10] R. Rasuli, *Anti fuzzy d-algebras and t-conorms*, 2^{th} National Conference on Soft Computing and Cognitive Science held by Conbad Kavous University, Factualy of Technology and Engineering Minutest, Iran, Minudasht, during April 18-19, 2023.

[11] R. Rasuli, *Fuzzy ideals of BCI-algebras with respect to t-norm*, Mathematical Analysis and its Contemporary Applications, **5(5)**(2023), 39-50.

[12] R. Rasuli, *Intuitionistic fuzzy complex subgroups with respect to norms(T and S)*, Journal of Fuzzy Extention and Application, **4(2)**(2023), 92-114.

[13] R. Rasuli, *Normality and translation of $IFS(G \times Q)$ under norms*, Notes on Intuitionistic Fuzzy Sets, **29(2)**(2023), 114-132.

[14] R. Rasuli, *Normality and translation of $IFS(G \times Q$ under norms*, International Conference on Intuitionistic Fuzzy Sets, 26-27 June 2023 Sofia, Bulgaria.

[15] R. Rasuli, *Some properties of fuzzy algebraic structures of QIFSN (G)*, International Conference on Computational Algebra, Computational Number Theory and Applications,CACNA2023, was held at the university of Kashan, Iran on 4-6 July 2023.

[16] A. Rosenfeld, *Fuzzy subgroups*, J. Mathl. Analy. Appl., **35**(1971), 512-517.

[17] T. K. Shinoj, A. Baby and J. J. Sunil, *On some algebraic structures of fuzzy multisets*, Ann. Fuzzy Math. Inform., **9**(1)(2015), 77-90.

[18] A. Syropoulos, *Mathematics of Multisets*, C. S. Calude et al. (Eds.): Multiset Processing, **2235**(2001), 347-358.

[19] A. Syropoulos, *Categorical models of multisets*, Romanian Journal Of Information Science and Technology, **6**(3-4)(2003), 393-400.

[20] L. A. Zadeh, *Fuzzy sets*, Inform. Control., **8**(1965), 338-353.

e-mail: Rasuli@pnu.ac.ir
e-mail: h.naraghi56@gmail.com
e-mail: bahman.taherkhani@pnu.ac.ir