Existence of a solution of the fractional integral system of equations via a measure of noncompactness

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Abstract

This paper studies the existence of a solution for a system of fractional integral equations using Darbo's fixed point theorem under the measure of noncompactness. Moreover, some tables and figures are presented to show the efficiency of our main results. In this paper, by applying the artificial small parameter method, we approximate the solution of one example.

Keywords: Fixed-point theorems, Integro-ordinary differential equations, Measures of noncompactness and condensing mappings.

AMS Mathematical Subject Classification [2010]: 13D45, 39B42

1 Introduction

Many papers successfully apply Darbo's fixed point theorem to study the existence of solutions for the functional integral equations formulated as a fixed point problem. For example in [4], the following system of integro-differential equations is studied

$$\begin{cases} \sigma(\varsigma) = f_1\Big(\varsigma, \sigma(\varsigma), \zeta(\varsigma), \int_p^{q_1(\varsigma)} g_1(\varsigma, s, \sigma(s), \zeta(s), \sigma^{'}(s), \zeta^{'}(s))\, ds\Big), \varsigma \in \mathbb{R}_+, \\ \zeta(\varsigma) = f_2\Big(\varsigma, \sigma(\varsigma), \zeta(\varsigma), \int_p^{q_2(\varsigma)} g_2(\varsigma, s, \sigma(s), \zeta(s), \sigma^{'}(s), \zeta^{'}(s))\, ds\Big), \varsigma \in \mathbb{R}_+. \end{cases} \tag{1}$$

In this article, we discuss the following fractional system of equations such that $\varsigma \in \mathbb{R}_+$

$$\begin{cases} \sigma(\varsigma) = f_1\Big(\varsigma, \sigma(\varsigma), \zeta(\varsigma), \frac{1}{\Gamma(\alpha_{1,1})} \int_{a_{1,1}}^{\varsigma} \frac{g^{'}_{1,1}(s)}{(g_{1,1}(\varsigma) - g_{1,1}(s))^{1-\alpha_{1,1}}}\, ds, ..., \frac{1}{\Gamma(\alpha_{1,n})} \int_{a_{1,n}}^{\varsigma} \frac{g^{'}_{1,n}(s)}{(g_{1,n}(t) - g_{1,n}(s))^{1-\alpha_{1,n}}}\, ds\Big), \\ \zeta(\varsigma) = f_2\Big(\varsigma, \sigma(\varsigma), \zeta(\varsigma), \frac{1}{\Gamma(\alpha_{2,1})} \int_{a_{2,1}}^{\varsigma} \frac{g^{'}_{2,1}(s)}{(g_{2,1}(\varsigma) - g_{2,1}(s))^{1-\alpha_{2,1}}}\, ds, ..., \frac{1}{\Gamma(\alpha_{2,n})} \int_{a_{2,n}}^{\varsigma} \frac{g^{'}_{2,n}(s)}{(g_{2,n}(t) - g_{2,n}(s))^{1-\alpha_{2,n}}}\, ds\Big), \end{cases} \tag{2}$$

in which $n \in \mathbb{N}$, $a_{i,j}, \alpha_{i,j} \in \mathbb{R}$, $i \in \{1, 2\}$, and $j \in \{1, ..., n\}$.

[1]speaker

2 Preliminaries

Definition 2.1. [4] A function $\varrho : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called the admissible-Darbo function, if it verifies one of the following conditions:
(D1)$\varrho^{-1}(\{0\}) = \{0\}$.
(D2) ϱ is a nondecreasing function and for $\varsigma \geq 0$, $\lim_{n\to\infty} \varrho^n(\varsigma) = 0$.
(D3) ϱ is an upper-semi continuous function and for $\varsigma > 0$, $\varrho(\varsigma) < \varsigma$.
(D4) For every $z > 0$, there exist $\delta > 0$ such that $\varrho(\varsigma) \leq z$ for $\varsigma \in [z, z+\delta]$.
Let us denote the family of admissible-Darbo functions with Υ.

Definition 2.2. [1] A mapping $\mu : M_Z \longrightarrow [0, +\infty]$ is a measure of noncompactness (MNC) in Z if it satisfies the following conditions:

(1) The family $ker\mu := \{J \in M_Z : \mu(J) = 0\}$ is nonempty subset of N_Z,

(2) $J_1 \subseteq J_2 \Longrightarrow \mu(J_1) \leq \mu(J_2)$,

(3) $\mu(\overline{J}) = \mu(J)$,

(4) $\mu(ConvJ) = \mu(J)$,

(5) $\mu(\alpha J_1 + (1-\alpha)J_2) \leq \alpha\mu(J_1) + (1-\alpha)\mu(J_2)$, for every $\alpha \in \big[0, 1\big]$,

(6) If $\{J_n\}$ is a sequence of closed sets from M_Z such that $J_{n+1} \subseteq J_n$ and $\lim_{n\to\infty} \mu(J_n) = 0$, then the set $J_\infty = \cap_{n=1}^{\infty} J_n$ is nonempty.

Theorem 2.3. *[4] Suppose Υ is a nonempty, convex, bounded and closed subset of a Banach space X. For each $i \in \{1, 2\}$ and $k \in \{1, ...n\}$ for some $n \in \mathbb{N}$, let $A_{i,k} : \Upsilon \times \Upsilon \longrightarrow \Upsilon$ be a compact and continuous operator, $\psi_i^k : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nondecreasing continuous function with $\psi_i^k(0) = 0$, and $T_i : \Upsilon \times \Upsilon \longrightarrow \Upsilon$ be an operator such that*

$$\| T_i(\sigma_1, \zeta_1) - T_i(\sigma_2, \zeta_2) \| \leq \varrho_i\Big(\max\{\|\sigma_1 - \sigma_2\|, \|\zeta_1 - \zeta_2\|\}\Big) + \sum_{k=1}^{n} \psi_i^k(\|A_{i,k}(\sigma_1, \zeta_1) - A_{i,k}(\sigma_2, \zeta_2)\|), \tag{3}$$

in which $\varrho_1, \varrho_2 \in \Upsilon$ are two same admissible-Darbo functions. Then there exist $x^*, y^* \in \Upsilon$ such that

$$\begin{cases} x^* = T_1(x^*, y^*), \\ y^* = T_2(x^*, y^*). \end{cases}$$

3 Main results

Theorem 3.1. *Consider the fractional system of equations* (2) *with the following conditions*

(A) For $i \in \{1, 2\}$, $j \in \{1, ..., n\}$, $\varsigma \in \mathbb{R}_+$, and $\sigma_i, \zeta_i, z_j \in \mathbb{R}$, we have

$$|f_i(\varsigma, \sigma_1, \zeta_1, z_1, ..., z_n) - f_i(\varsigma, \sigma_2, \zeta_2, z_1, ..., z_n)| \leq \varrho_i\Big(\max\{\|\sigma_1 - \sigma_2\|, \|\zeta_1 - \zeta_2\|\}\Big).$$

(B) There exists a positive constant r_0 such that for $i \in \{1, 2\}$,

$$\varrho_i(r_0) + f_0^i < r_0, \tag{4}$$

in which

$$f_0^i := \sup\{|f_i(\varsigma,0,0,\frac{1}{\Gamma(\alpha_{i,1})}\int_{a_{i,1}}^{\varsigma}\frac{g'_{i,1}(s)}{\big(g_{i,1}(\varsigma)-g_{i,1}(s)\big)^{1-\alpha_{i,1}}}ds,...,\frac{1}{\Gamma(\alpha_{i,n})}\int_{a_{i,n}}^{\varsigma}\frac{g'_{i,n}(s)}{\big(g_{i,n}(\varsigma)-g_{i,n}(s)\big)^{1-\alpha_{i,n}}}ds)|:\varsigma\in\mathbb{R}_+\}. \tag{5}$$

Proof. Consider the operator T_i on the space $C(\mathbb{R}_+)$ for $i\in\{1,2\}$, as follows:

$$T_i(\sigma,\zeta)(\varsigma)=f_i\Big(\varsigma,\sigma(\varsigma),\zeta(\varsigma),\frac{1}{\Gamma(\alpha_{i,1})}\int_{a_{i,1}}^{\varsigma}\frac{g'_{i,1}(s)}{\big(g_{i,1}(\varsigma)-g_{i,1}(s)\big)^{1-\alpha_{i,1}}}\,ds,...,\frac{1}{\Gamma(\alpha_{i,n})}\int_{a_{i,n}}^{\varsigma}\frac{g'_{i,n}(s)}{\big(g_{i,n}(\varsigma)-g_{i,n}(s)\big)^{1-\alpha_{i,n}}}\,ds\Big).$$

Firstly, for every $\sigma,\zeta\in B_{r_0}=\{y\in C(\mathbb{R}_+):\|y\|_{C(\mathbb{R}_+)}\le r_0\}$, we have to prove $T_i(\sigma,\zeta)\in B_{r_0}$. Now, for $\varsigma\in\mathbb{R}_+$, we calculate as follows:

$$\begin{aligned}|T_i(\sigma,\zeta)(\varsigma)|=\Big|&f_i\Big(\varsigma,\sigma(\varsigma),\zeta(\varsigma),\frac{1}{\Gamma(\alpha_{i,1})}\int_{a_{i,1}}^{\varsigma}\frac{g'_{i,1}(s)}{\big(g_{i,1}(\varsigma)-g_{i,1}(s)\big)^{1-\alpha_{i,1}}}\,ds,...,\frac{1}{\Gamma(\alpha_{i,n})}\int_{a_{i,n}}^{\varsigma}\frac{g'_{i,n}(s)}{\big(g_{i,n}(\varsigma)-g_{i,n}(s)\big)^{1-\alpha_{i,n}}}\,ds\Big)\\
&-f_i\big(\varsigma,0,0,\frac{1}{\Gamma(\alpha_{i,1})}\int_{a_{i,1}}^{\varsigma}\frac{g'_{i,1}(s)}{\big(g_{i,1}(\varsigma)-g_{i,1}(s)\big)^{1-\alpha_{i,1}}}\,ds,...,\frac{1}{\Gamma(\alpha_{i,n})}\int_{a_{i,n}}^{\varsigma}\frac{g'_{i,n}(s)}{\big(g_{i,n}(\varsigma)-g_{i,n}(s)\big)^{1-\alpha_{i,n}}}\,ds\big)\\
&+f_i\big(\varsigma,0,0,\frac{1}{\Gamma(\alpha_{i,1})}\int_{a_{i,1}}^{\varsigma}\frac{g'_{i,1}(s)}{\big(g_{i,1}(\varsigma)-g_{i,1}(s)\big)^{1-\alpha_{i,1}}}\,ds,...,\frac{1}{\Gamma(\alpha_{i,n})}\int_{a_{i,n}}^{\varsigma}\frac{g'_{i,n}(s)}{\big(g_{i,n}(\varsigma)-g_{i,n}(s)\big)^{1-\alpha_{i,n}}}\,ds\big)\Big|\\
&\le\varrho_i(r_0)+f_0^i.\end{aligned}$$

By applying conditions (A) and (B), we obtain $T_i(\sigma,\zeta)\in B_{r_0}$. In what follows, we study the continuity of T_i on $B_{r_0}\times B_{r_0}$. To do this, suppose $\{\sigma_n\}$ and $\{\zeta_n\}$ as sequences in B_{r_0} that converge to $\sigma\in B_{r_0}$ and $\zeta\in B_{r_0}$, respectively. For every $\varepsilon>0$, there exists $N>0$ such that $\|\sigma_n-\sigma\|_{C(\mathbb{R}_+)},\|\zeta_n-\zeta\|_{C(\mathbb{R}_+)}\le\varepsilon$. Thus, for all $n>N$ and $\varsigma\in\mathbb{R}_+$, we get

$$|T_i(\sigma_n,\zeta_n)(\varsigma)-T_i(\sigma,\zeta)(\varsigma)|\le\varrho_i\Big(\max\{\|\sigma_n-\sigma\|,\|\zeta_n-\zeta\|\}\Big).$$

Then the existence of a solution is derived from Theorem 2.3.

□

Example 3.2. The following fractional integral system of equations has at least one solution

$$\begin{aligned}\sigma(\varsigma)=&\frac{e^{-\varsigma}}{12}+\frac{\arctan(\frac{\sqrt{\varsigma}}{\cosh(\varsigma)})}{12}\sin(\zeta(\varsigma))+\frac{1}{\Gamma(\frac13)}\int_1^{\varsigma}\frac{e^{-s}\big(s-2\big)}{\big((1-\varsigma)e^{-\varsigma}-(1-s)e^{-s}\big)^{\frac23}}\,ds\\
&+\frac{\sin(\varsigma)}{\Gamma(\frac17)}\int_2^{\varsigma}\frac{e^{-s}\big(s-3\big)}{\big((2-\varsigma)e^{-\varsigma}-(2-s)e^{-s}\big)^{\frac67}}\,ds,\\
\zeta(\varsigma)=&\frac{\operatorname{sech}(\varsigma)}{\varsigma^7+e^{\varsigma}}+\frac{\ln(1+|\sigma(\varsigma)|)}{\sin(\varsigma)+12}+\frac{1}{\Gamma(\frac13)}\int_2^{\varsigma}\frac{-2s(s^4-8s^2-1)}{(s^4+1)^2\Big(\frac{\varsigma^2-4}{\varsigma^4+1}-\frac{s^2-4}{s^4+1}\Big)^{\frac23}}\,ds\\
&+\frac{1}{\Gamma(\frac13)}\int_{\sqrt3}^{\varsigma}\frac{-2s(s^4-6s^2-1)}{(s^4+1)^2\Big(\frac{\varsigma^2-3}{\varsigma^4+1}-\frac{s^2-3}{s^4+1}\Big)^{\frac23}}\,ds.\end{aligned}$$

Proof. If we get

$$
\begin{aligned}
f_1(\varsigma,\sigma,\zeta,z_1,z_2) &= \frac{e^{-\varsigma}+\sin(\zeta)\arctan(\frac{\sqrt{\varsigma}}{\cosh(\varsigma)})}{12}+z_1+\sin(\varsigma)z_2,\\
f_2(\varsigma,\sigma,\zeta,z_1,z_2) &= \frac{\operatorname{sech}(\varsigma)}{\varsigma^7+e^{\varsigma}}+\frac{\ln(1+|\sigma|)}{\sin(\varsigma)+12}+z_1+z_2,\\
g_{1,1}(\varsigma) &= (1-\varsigma)e^{-\varsigma},\\
g_{1,2}(\varsigma) &= (2-\varsigma)e^{-\varsigma},\\
g_{2,1}(\varsigma) &= \frac{\varsigma^2-4}{\varsigma^4+1},\\
g_{2,2}(\varsigma) &= \frac{\varsigma^2-3}{\varsigma^4+1}.
\end{aligned}
$$

Now, due to condition (B) and Theorem (3.1), if $\varrho_1(\sigma)=\frac{\pi\sigma}{24}$ and $\varrho_2(\sigma)=\frac{\ln(1+\sigma)}{12}$, the existence of at least one solution is derived for $r_0=12$. □

4 Numerical method

We provide an example of functional integro-differential equations of the form (2) for which the assumptions of Theorem 3.1 are satisfied. Moreover, applying the artificial small parameter method [11, 7], which is a particular case of homotopy analysis method [10, 9] and also is equivalent to the Adomian decomposition method [8, 6], we find an approximation of the solution which converges. The tables and figures show the effectiveness of our method.

Example 4.1. Consider the following equation.

$$
\begin{aligned}
\sigma(\varsigma) =& \frac{\sin(\zeta(\varsigma))}{\varsigma^4+10}-\frac{1}{6\Gamma(\frac{1}{3})}\int_1^{\varsigma}\frac{e^{-s}(s-2)}{\big((1-\varsigma)e^{-\varsigma}-(1-s)e^{-s}\big)^{\frac{2}{3}}}\,ds,\\
\zeta(\varsigma) =& \frac{\sigma(\varsigma)}{\varsigma^8+18}-\frac{1}{60\Gamma(\frac{1}{3})}\int_2^{\varsigma}\frac{-2s(s^4-8s^2-1)}{(s^4+1)^2\Big(\frac{\varsigma^2-4}{t^4+1}-\frac{s^2-4}{s^4+1}\Big)^{\frac{2}{3}}}\,ds.
\end{aligned}
\tag{6}
$$

that is an example for equation (2) with $f_1(\varsigma,\sigma,\zeta,z):=\frac{\sin(\zeta)}{\varsigma^4+10}+\frac{z}{6}$, $f_2(\varsigma,\sigma,\zeta,z):=\frac{\sigma}{\varsigma^8+18}+\frac{z}{60}$, $g_1:=(1-\varsigma)e^{-\varsigma}$, $g_2:=\frac{\varsigma^2-4}{\varsigma^4+1}$. Thus assumptions are verified for each $0.2\le r_0$. For solving this equation we consider the following equation:

$$
\begin{aligned}
\sum_{j=0}^{\infty}\rho^j v_j(\varsigma) =& \frac{\sin(\sum_{j=0}^{\infty}\rho^j w_j(\varsigma))}{\varsigma^4+10}-\frac{1}{6\Gamma(\frac{1}{3})}\int_1^{\varsigma}\frac{e^{-s}(s-2)}{\big((1-\varsigma)e^{-\varsigma}-(1-s)e^{-s}\big)^{\frac{2}{3}}}\,ds,\\
\sum_{j=0}^{\infty}\rho^j w_j(\varsigma) =& \frac{\sum_{j=0}^{\infty}\rho^j v_j(\varsigma)}{\varsigma^8+18}-\frac{1}{60\Gamma(\frac{1}{3})}\int_2^{\varsigma}\frac{-2s(s^4-8s^2-1)}{(s^4+1)^2\Big(\frac{\varsigma^2-4}{\varsigma^4+1}-\frac{s^2-4}{s^4+1}\Big)^{\frac{2}{3}}}\,ds.
\end{aligned}
$$

The parameter ρ is called the artificial small parameter. The approximate solution of $\sigma(\varsigma)$ and $\zeta(\varsigma)$ are $Q(\varsigma,\rho)=\sum_{j=0}^{\infty}\rho^j v_j(\varsigma)$ and $\Theta(\varsigma,\rho)=\sum_{j=0}^{\infty}\rho^j w_j(\varsigma)$, respectively. If ρ increases from 0 to 1, then $Q(\varsigma,1)$ and

$\Theta(\varsigma,1)$ are equivalent to $\sigma(\varsigma)$ and $\zeta(\varsigma)$, respectively. We can write

$$\sin(\Theta(\varsigma,\rho)) = \sin(\sum_{j=0}^{\infty}\rho^j w_j(\varsigma)) = \sum_{j=0}^{\infty}\rho^j A_j(\varsigma), \tag{7}$$

Differentiating both sides of the above expression n times with respect ρ and then setting $\rho=0$, we get

$$A_n(\varsigma) = \frac{1}{n!}\frac{\partial^n}{\partial\rho^n}\sin\Big(\sum_{i=0}^{\infty}\rho^i\big(\Theta(\varsigma,\rho)\big)\Big)\Big|_{\rho=0},$$

Therefore, we have

$$\begin{aligned}\sum_{j=0}^{\infty}\rho^j v_j(\varsigma) =& \frac{\rho\sum\limits_{j=0}^{\infty}\rho^j A_j(\varsigma)}{\varsigma^4+10} + \frac{((1-\varsigma)e^{-\varsigma})^{\frac{1}{3}}}{2\Gamma(\frac{1}{3})},\\ \sum_{j=0}^{\infty}\rho^j w_j(\varsigma) =& \frac{\sum\limits_{j=0}^{\infty}\rho^j v_j(\varsigma)}{\varsigma^8+18} + \frac{1}{20\Gamma(\dfrac{1}{3})}(\frac{\varsigma^2-4}{\varsigma^4+1})^{\frac{1}{3}}.\end{aligned} \tag{8}$$

If we compare the two sides of (8) with respect to the powers of ρ, we can get v_0, w_0, v_1 and w_1 as follows,

$$\begin{aligned}v_0(\varsigma) =& \frac{((1-\varsigma)e^{-\varsigma})^{\frac{1}{3}}}{2\Gamma(\dfrac{1}{3})},\\ w_0(\varsigma) =& \frac{((1-\varsigma)e^{-\varsigma})^{\frac{1}{3}}}{2\Gamma(\dfrac{1}{3})(\varsigma^8+18)} + (\frac{\varsigma^4-4}{\varsigma^4+1})^{\frac{1}{3}} \times \frac{1}{20\Gamma(\dfrac{1}{3})},\\ v_1(\varsigma) =& \frac{\sin(w_0(\varsigma))}{\varsigma^4+10},\\ w_1(\varsigma) =& \frac{v_1(\varsigma)}{\varsigma^8+18}.\end{aligned}$$

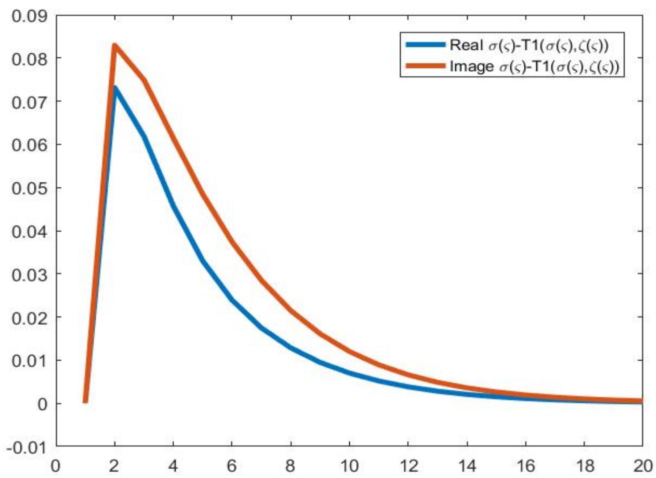

Fig.1

i	ς	$Real(\sigma(\varsigma) - T_1(\sigma,\zeta)(\varsigma))$	$Imag(\sigma(\varsigma) - T_1(\sigma,\zeta)(\varsigma))$
1	1	4.64×10^{-6}	8.04×10^{-6}
2	2	7.32×10^{-2}	8.3×10^{-2}
3	3	6.18×10^{-2}	7.49×10^{-2}
4	4	4.57×10^{-2}	6.14×10^{-2}
5	10	7×10^{-3}	1.20×10^{-2}
6	11	5.2×10^{-3}	8.9×10^{-3}

Table 1: Absolute errors of $T_1(\sigma,\zeta)$.

i	ς	$Real(\zeta(\varsigma) - T_2(\sigma,\zeta)(\varsigma))$	$Imag(\zeta(\varsigma) - T_2(\sigma,\zeta)(\varsigma))$
1	1	1.94×10^{-19}	3.36×10^{-19}
2	2	1.66×10^{-2}	0
3	3	1.09×10^{-2}	0
4	4	1.18×10^{-2}	0
5	10	1.47×10^{-2}	0
6	11	1.49×10^{-2}	0

Table 2: Absolute errors of $T_2(\sigma,\zeta)$.

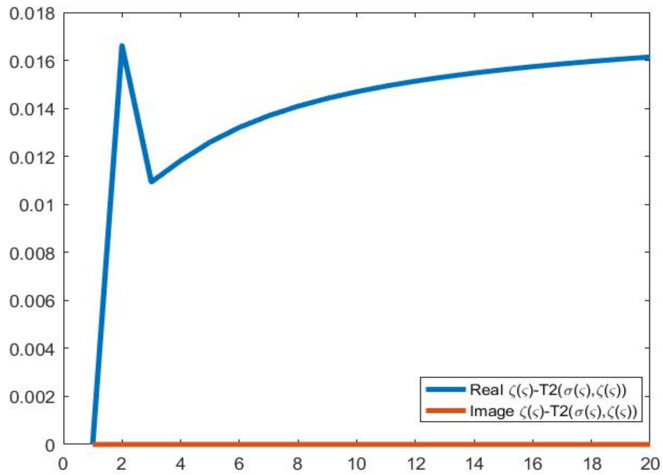

Fig.2

We approximate $\sigma(\varsigma) \approx v_0(\varsigma) + v_1(\varsigma)$ and $\zeta(\varsigma) \approx w_0(\varsigma) + w_1(\varsigma)$.

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