Semisimple-Intersection Graphs of Ideals of Rings

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ABSTRACT

In this paper, a new kind of graph on a ring is introduced and investigated. Let R be a ring with unity. Semisimple-intersection graph of a ring R, denoted by $G_S(R)$, is an undirected simple graph with all nonzero ideals of R as vertices and two distinct vertices I and J are adjacent if and only if $I \cap J$ is a nonzero semisimple ideal of R. In this article, we investigate the basic properties of these graphs to relate the combinatorial properties of $G_S(R)$ to the algebraic properties of the ring R. We determine the diameter and the girth of $G_S(R)$. We obtain some results for connectedness and bipartiteness of these graphs, as well as give a formula to compute the clique numbers of $G_S(R)$. We observed that the graph $G_S(R)$ is complete if and only if every proper ideal of R either simple or semisimple and every pair of ideals in R have non-zero intersection.

KEYWORDS: Semisimple-intersection graph, rings, ideals, clique number, girth, diameter.

1 INTRODUCTION

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in the last decade. Bosak in 1964 [7] introduced the concept of the intersection graph of semigroups. In 1988, Istvan Beck [5] proposed the study of commutative rings by representing them as graphs, named zero divisor graph. These zero divisor graphs marked the beginning of an approach to studying commutative rings with graphs. Similarly, there is several graphs assigned to modules and semirings [2-4]. In 2009, the intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [8]. The intersection graph of ideals of a ring R is a graph having the set of all nonzero ideals of R as its vertex set and two distinct vertices I and J are adjacent if and only if $I \cap J$ is non-trivial (non-zero). The intersection graph of ideals was studied by a large number of mathematicians, in which they obtained many of its properties and linked them to the properties of rings as demonstrated in [1, 9, 11-14]. In this paper, we introduce semisimple-intersection graphs of ideals of rings as the following.

Definition 1.1: Let R be a ring with unity. The semisimple-intersection graph of R, denoted by $G_S(R)$, is defined to be a simple graph whose vertices are the nonzero ideals of R, and two vertices I and J are adjacent, and we write $I - J$, if and only if $I \cap J$ is a nonzero semisimple ideal.

For example, consider the ring $\mathbb{Z}_4$. The nonzero ideals of $\mathbb{Z}_4$ are $\mathbb{Z}_4$ and $2\mathbb{Z}_4 \cong \mathbb{Z}_2$. Obviously, $G_S(\mathbb{Z}_4)$ is $\mathbb{Z}_4 - 2\mathbb{Z}_4$. Clearly, the semisimple-intersection graph is a subgraph of the intersection graph of ideals of R. We study several properties of this graph such as connectedness, regularity, girth, cliques, the "bipartite" property. Some preliminaries from ring theory and graph theory are listed in Section 2. In Section 3, we show that the girth of $G_S(R)$ equals one of the three values 3, 4, or ∞. We also show that the diameter of

$G_S(R)$ does not exceed 4 and never equals 3. In Section 4, we give explicit formulas to compute the clique number of $G_S(R)$. Besides, a necessary and sufficient conditions for $G_S(R)$ to be bipartite are provided. We give many examples to illustrate the concepts discussed here, and provide counterexamples for expected results.

2 PRELIMINARIES

This section presents a fast review of rings and graphs that are important in this work. All notions of graph theory presented here could be found in [6], also, all notions of ring theory presented here could be found in [10]. In this paper, all rings are assumed to be nonzero rings with unity and are not necessarily commutative. Besides, the ideals are considered to be left ideals. We first start with some preliminaries from Ring Theory.

Definition 2.1: An ideal I of a ring R is said to be simple (or minimal) if I and $\{0\}$ are the only ideals included in I.

Definition 2.2: The direct sum of simple ideals of a ring R is called a semisimple ideal. We call each simple ideal in the decomposition of a semisimple ideal, a component.

Obviously, a simple ideal is semisimple with one component. On the other hand, every ideal of a semisimple ideal is semisimple.

Definition 2.3: The socle of R, denoted by $Soc(R)$, is defined to be the sum (precisely, the direct sum) of all nonzero simple ideals of R. If $R = Soc(R)$, we call R a semisimple ring.

Definition 2.4: A proper ideal I of a ring R is said to be maximal if I is not contained in another proper ideal of R.

Definition 2.5: A ring R is said to be Artinian if it satisfies the descending chain condition on ideals.

Theorem 2.6: A ring R is Artinian if and only if every nonzero ideal contains a nonzero simple ideal.

As a direct corollary from Theorem 2.6, semisimple rings are Artinian.

Next, we turn to preliminaries from Graph theory concerning undirected graphs. In what follows, G indicates an undirected graph. The number of vertices of G is called the order the graph G. The set of vertices of G is denoted by $V[G]$. If two vertices u and v are adjacent, we express that symbolically by $u - v$.

Definition 2.7: Let v be a vertex in G. The neighborhood $N(v)$ of v is the set of all vertices adjacent to v, i.e., the set of all vertices each of which is linked to v by an edge.

If G is a semisimple undirected graph, then $v < N(v)$. If $N(v) = \emptyset$, then v is an isolated vertex.

Definition 2.8: The degree of a vertex v of G is the number of edges incident to v, i.e., going out of v. The degree of v is denoted by $\deg_G(v)$ (or $\deg(v)$ if there is no confusion with the underlined graph). The minimum of the degrees of the vertices is denoted by $\delta(G)$, while the maximum of the degrees of the vertices is called the degree of the graph and is denoted by $\Delta(G)$.

When G is a simple graph, then $\deg(v) = |N(v)|$, where $|N(v)|$ means the cardinality of $N(v)$. So, v is isolated if and only if $\deg(v) = 0$.

Definition 2.9: A graph whose vertices have equal degrees is called a regular graph. A graph G is regular if and only if $\Delta(G) = \delta(G)$.

Definition 2.10: Let v and u be two vertices of G. The length of a path between v and u is the number of edges forming the path. The distance $d(u, v)$ between v and u is the length of a shortest path between them. The diameter of G, denoted by $\mathrm{diam}(G)$, is defined to be the supremum of the set $\{d(u, v) : u, v \in V[G]\}$.

Definition 2.11: A graph G is path connected if there is a path between any two vertices of G.

Definition 2.12: A graph is said to be complete if it is a simple graph and every pair of vertices are adjacent. The complete graph on n vertices is denoted by K_n.

Definition 2.13: A subgraph of G which is a complete graph is called a clique of G. The order of a largest clique (i.e., a clique with the largest number of vertices) is called the clique number of G and it is denoted by $\omega(G)$.

Definition 2.14: By the girth of G, we mean the length of a shortest cycle in G. The girth of G is denoted by $\mathrm{gr}(G)$. If G has no cycles, then we write $\mathrm{gr}(G) = \infty$.

Definition 2.15: A simple graph G is named bipartite if we can partition $V[G]$ into two disjoint nonempty subsets (each subset is called a part) such that the vertices belonging to the same subset are not adjacent to each other. A complete bipartite graph is a bipartite graph where each vertex in one part is adjacent to each vertex in the other part. A complete bipartite graph is denoted by $K_{m,n}$, where m is the cardinality of one part and n is the cardinality of the other part.

3 CONNECTIVITY, DIAMETER AND GIRTH OF $G_S(R)$

In this section, we study the connectivity and girth of $G_S(R)$. Also, we determine the diameter of $G_S(R)$. In addition, we give the necessary and sufficient conditions for $G_S(R)$ to be a complete graph.

Recall that a null graph is a graph whose vertices are not adjacent to each other (i.e., edgeless graph).

Theorem 3.1: The graph $G_S(R)$ is a null graph if and only if R is a simple ring or R has no nonzero simple ideals.

***Proof*:** Assume $G_S(R)$ is a null graph. Suppose for contrary that R is not simple and it contains a nonzero simple ideal I, so I is semisimple. Then $R - I$ and hence $G_S(R)$ is not null, which is a contradiction to the hypothesis "$G_S(R)$ is a null graph". The `converse is easy. □

Example 3.2: $G_S(\mathbb{Z})$ and $G_S(\mathbb{Z}_2)$ are null.

Remark 3.3: In $G_S(R)$, $I - R$ if and only if I is a nonzero semisimple (or simple) ideal of R. So the subgraph consisting of R with all nonzero simple ideals of R is a star graph with center R, and hence $\deg(R)$ equals the number of nonzero simple ideals ideals of R. Thus, if R is semisimple with n components, then $\deg(R) = n$. On the other hand, if I is a nonzero simple ideal of R and J is an ideal of R, then $I - J$ if and only if $I \subsetneq J$. Moreover, every pair of different nonzero simple ideals is not adjacent, or equivalently, the subgraph of $G_S(R)$ consisting of nonzero simple ideals is a null graph.

Next, we study the cycles of $G_S(R)$. We show that the cycles with length at least 4 has only two special patterns. Afterward, we demonstrate that the girth of $G_S(R)$ is either ∞, 3, or 4.

Theorem 3.4: The graph $G_S(R)$ has a cycle of length 3 if and only if $G_S(R)$ contains at least two adjacent non-semisimple ideals.

***Proof*:** Assume that the graph $G_S(R)$ has a cycle of length 3, say $I - J - K - I$. By Remark 3.3, at most one of the vertices is semisimple. So, at least two of the vertices in this cycle are adjacent non-semisimple ideals of R. For the converse, let I and J be two different nonzero non-semisimple ideals such that $I \cap J$ is nonzero semisimple. Then we have the cycle $I - J - I \cap J - I$. □

Theorem 3.5: If the graph $G_S(R)$ has a cycle, then either $\mathrm{gr}(G_S(R)) = 3$ or $\mathrm{gr}(G_S(R))$ is even.

***Proof*:** Assume that $G_S(R)$ has the cycle $I_1 - I_2 - \ldots - I_n - I_1$, and $n > 3$, with the least length. By Remark 3.3 and Theorem 3.4, if two vertices in the cycle are adjacent, then one of them is nonzero semisimple and the other is non-semisimple. In other words, the vertices of the cycle alternate between semisimple and non-semisimple ideals. Without loss of generality, assume I_1 is semisimple. Then I_2 is not semisimple, I_3 is semisimple, I_4 is not semisimple, and so on. Thus, for each $1 \le k \le n$, I_k is semisimple if k is odd, and I_k is not semisimple if k is even. If n is odd, then I_n is semisimple and this implies $I_{n+1} = I_1$ is not semisimple which number of contradicts the assumption that I_1 is semisimple. We conclude that n is even. Since the length of a cycle equals n, the its vertices, we get that the length of the cycle is even. As a result, any cycle in $G_S(R)$ has either a length of 3 or has an even length. So, $\mathrm{gr}(G_S(R)) = 3$ or $\mathrm{gr}(G_S(R))$ is even. □

Corollary 3.6: Let R be a semisimple ring. If R has more than two components, then $\mathrm{gr}(G_S(R)) = 3$. Otherwise, $\mathrm{gr}\big(G_S(R)\big) = \infty$.

***Proof*:** Assume R has at least three components. Let $I \oplus J \oplus K$ be a semisimple ideal of R. Then $I \oplus J$ and $J \oplus K$ are nonzero different non-semisimple ideals whose intersection is the simple ideal J. By Theorem 3.4, $G_S(R)$ has a cycle of length 3, and hence $\mathrm{gr}(G_S(R)) = 3$. The rest of the proof is easy. □

Remark 3.7: The proof of Theorem 3.5 displays a technique to build a cycle of a shortest length more than 3 as illustrated in the next example.

Example 3.8: Let $R = I \oplus J \oplus K \oplus T$ be a semisimple ring with 4 components. Then $P_1: I - I \oplus J \oplus K - J - I \oplus J - I$ and $P_2 : I \oplus T - J \oplus T - I \oplus J - I \oplus T$ are two different cycles of length 4. Notice that P_1 was constructed using the technique applied in the proof of Theorem 3.5; that is the vertices of P_1 alternate between semisimple and non-semisimple ideals. However, P_2 was built in different way where all its vertices are not simple ideals.

Example 3.9: Given the ring $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, the cycle $\mathbb{Z}_4 \oplus \mathbb{Z}_2 - 0 \oplus \mathbb{Z}_2 - \mathbb{Z}_2 \oplus \mathbb{Z}_2 - \mathbb{Z}_2 \oplus 0 - \mathbb{Z}_4 \oplus \mathbb{Z}_2$ is a shortest cycle in $G_S(\mathbb{Z}_4 \oplus \mathbb{Z}_2)$. Thus, $\mathrm{gr}(G_S(\mathbb{Z}_4 \oplus \mathbb{Z}_2)) = 4$.

The next theorem is a natural extension of the preceding work.

Theorem 3.10: Let R be a ring with the graph $G_S(R)$. Then $\mathrm{gr}(G_S(R)) \in \{3, 4, \infty\}$.

***Proof*:** According to Theorem 3.4, if R has no nonzero semisimple ideals, or R is simple, then $\mathrm{gr}(G_S(R)) = \infty$. Take now that R includes exactly one nonzero semisimple ideal I. If I is a maximal ideal (which is possible if all elements of R outside I are units), then $G_S(R)$ is just $R - I$ and henceforth $\mathrm{gr}(G_S(R)) = \infty$. If I is not a maximal ideal, then I is included in proper non-semisimple ideals. At this point, if I is included in a unique proper non-semisimple ideal J, then $R - I$ and $J - I$ are the only paths in $G_S(R)$ and so $\mathrm{gr}(G_S(R)) = \infty$. However, if I is included in more than one proper non-semisimple ideal, then either there are two proper non-semisimple ideals intersect at I, which implies by Theorem 3.4 that $\mathrm{gr}(G_S(R)) = 3$, or no two proper non-semisimple ideals intersect at I, which implies $G_S(R)$ is a star graph with center I and thus $\mathrm{gr}(G_S(R)) = \infty$. Next, suppose that R has at least two different nonzero semisimple ideals. Let I and J be two different nonzero semisimple ideals. We distinguish between two cases. In the first case, suppose $R = I \oplus J$. By Corollary 3.6, $\mathrm{gr}(G_S(R)) = \infty$. In the second case, suppose that $R \neq I \oplus J$. Then, the cycle $R - I - I \oplus J - J - R$ is a cycle of length 4. The latter cycle has the least length unless R possesses two different non-semisimple ideals whose intersection is a nonzero semisimple ideal, which yields by Theorem 3.4 the existence of a cycle of length 3. □

Lemma 3.11: The distance between two nonzero simple (or semisimple) ideals in $G_S(R)$ is 2.

***Proof*:** Let I and J be two nonzero simple ideals of R. Then I and J are not adjacent, however, they are connected by the path $I - R - J$ which is a shortest path between I and J. Thus, $d(I, J) = 2$. Now, assume I and J are two semisimple ideals of R. Then either the path $I - R - J$ or the path $I - I \cap J - J$ which is a shortest path between I and J. This completed the proof. □

Theorem 3.12: The graph $G_S(R)$ is path connected if and only if R is an Artinian ring.

***Proof*:** Suppose $G_S(R)$ is path connected and R is not simple. Let I be a nonzero ideal of R. Now, there is a path from I to R. If the path has a length equal to 1, then $I - R$ and hence I is a semisimple ideal. If the path has a length of at least 2, then there exists an ideal J, R such that $J - I$. Thus, $I \cap J$ is a nonzero semisimple ideal contained in I. Thus there is K is a nonzero simple ideal with $K \subseteq I \cap J \subseteq I$. As each nonzero ideal contains a nonzero simple ideal, we conclude that R is Artinian.

For the reverse, assume R is an Artinian ring. Let I and J be two nonzero ideals. Now, there is nonzero simple ideals K and L with $K \subseteq I$ besides $L \subseteq J$. The path $I - K - K \oplus L - L - J$ connects I and J. So, $G_S(R)$ is path connected. □

Corollary 3.13: If R is Artinian, and I and J are nonzero ideals, then $1 \leq d(I, J) \leq 4$.

***Proof*:** Suppose R is Artinian, and I and J are nonzero ideals. By Theorem 3.12, $d(I, J) \geq 1$. We consider different cases. In the first case, assume that both ideals are simple, by Lemma 3.11, we have $d(I, J) = 2$. In the second case, assume exactly one of them, say I, is simple. If $I \subset J$, then $I - J$ and therefore $d(I, J) = 1$. Though, if $I \nsubseteq J$, then $I \cap J = 0$. Let $0 \neq K \subsetneq J$ be a simple ideal. Then, $I - I \oplus K - J$ is the shortest path between I and J. So, $d(I, J) = 2$. In the third case, assume neither of I and J is simple. If $I \cap J$ is nonzero semisimple, then $I - J$ and thus $d(I, J) = 1$. If $I \cap J$ is not semisimple, then it contains a nonzero simple ideal K. The path $I - K - J$ is the shortest path between I and J and hence $d(I, J) = 2$. If $I \cap J = 0$, then the path $I - K - K \oplus L - L - J$ is the shortest path between I and J, where K and L are nonzero simple ideals of R contained in I and J, respectively. So, $d(I, J) = 4$. □

Recall that a path component of a graph is the largest path connected subgraph (i.e., a connected subgraph that is not a subgraph of another connected subgraph). A graph may have more than one path component. A graph is connected if and only if the path component is unique. In addition, the diameter of a graph equals the supremum of the diameter of its path components.

Theorem 3.14: In $G_S(R)$, all the vertices with positive degree lie in one path component which is the path component containing R. The rest of components are isolated vertices.

***Proof*:** A discussion analogous to that in the proof of Corollary 3.13 leads us to the fact that any vertex of positive degree is connected to R by a path of length less than 3. □

Definition 3.15: The component of R mentioned in Theorem 3.14 is named the R-component of $G_S(R)$.

According to Theorem 3.14, we can, in many situations, identify $G_S(R)$ with its R-component, since we can carry the discussion of a disconnected $G_S(R)$ to its R- component.

The proof of the next corollary follows easily from Corollary 3.13 and Theorem 3.14.

Corollary 3.16: $1 \leq \operatorname{diam}(G_S(R)) \leq 4$. Besides, $\operatorname{diam}(G_S(R)) = 1$ if and only if the R-component of $G_S(R)$ is a complete graph consisting of at least two vertices.

Theorem 3.17: The graph $G_S(R)$ is complete if and only if every proper ideal of R either simple or semisimple and every pair of ideals in R have non-zero intersection.

***Proof*:** Assume that $G_S(R)$ is complete. Let I be an ideal of R. By completeness, we have $I - R$ and thus I is semisimple. Let I and J be an ideals of R. Thus, $I \cap J$ is semisimple, and hence $I \cap J \neq (0)$. For the converse, assume that I and J be an ideals of R. By assumption I and J are simple or semisimple ideals. Thus, $I \cap J$ is semisimple ideal. Hence, the graph $G_S(R)$ is complete. □

The next corollary is a straight forward result from Theorem 3.17.

Corollary 3.18: If R is a semisimple ring, then $G_S(R)$ is not complete.

Example 3.19: $G_S(\mathbb{Z}_4)$ is K_2, while $G_S(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ is not complete.

We have seen so far that if $G_S(R)$ is not a null graph, then $1 \leq \operatorname{diam}(G_S(R)) \leq 4$. Surprisingly, $\operatorname{diam}(G_S(R))$ never reaches 3 as demonstrated in the next theorem.

Theorem 3.20: We have $\operatorname{diam}(G_S(R)) \neq 3$.

***Proof*:** Suppose $G_S(R)$ is not a null graph. By contrary way. Assume there exist two nonzero ideals of R such that $d(I,J) = 3$. So that the shortest path connecting I and J has a length equal to 3. Let $I - K - L - J$ be such a shortest path. Thus we have I is not adjacent to J, J is not adjacent to K, I is not adjacent to L, and at least one of K and L is not simple. We have $I \cap K$ and $J \cap L$ are nonzero semisimple ideals. Also, $I \cap J \cap L = 0$ (because if not, then the simplicity of $J \cap L$ implies $I \cap J \cap L = J \cap L$, which in turn implies $J \cap L \subseteq I$, which yields that I and J are connected by the path $I - (J \cap L) - J$ of length 2. This contradicts that $d(I,J) = 3$. Similarly, we obtain that $J \cap I \cap K = 0$. Now, let $T = (I \cap K) \oplus (J \cap L)$. Then $I \cap T = I \cap K$ and $J \cap T = J \cap L$ which are nonzero semisimple ideals. So, we have the path $I - T - J$ is a path connecting I and J of length equal to 2, which is a illogicality to the assumption $d(I,J) = 3$. In conclusion, we get that $d(I,J) \neq 3$. □

4 THE BIPARTITE PROPERTY AND CLIQUES OF $G_S(R)$

In this section, we study the clique number of $G_S(R)$. Also, here the conditions which make $G_S(R)$ bipartite are studied. Since the null graph is obviously bipartite, the importance here in our consideration is the non-null $G_S(R)$. Recall that $G_S(R)$ is not null if and only if R possesses a proper nonzero simple ideal.

We begin with the following corollary which is an outcome of Section 3.

Corollary 4.1: If $3 < \mathrm{gr}(G_S(R)) < \infty$, then $G_S(R)$ is bipartite.

***Proof*:** Assume $3 < \mathrm{gr}(G_S(R)) < \infty$. Following the similar argument of the proof of Theorem 3.5, we get that no nonzero simple ideals are adjacent to each other and no non-simple ideals are adjacent to each other. Thus, the graph $G_S(R)$ is bipartite with parts W_1 consisting of all nonzero simple ideals of R, and W_2 consisting of all non-simple ideals of R. □

Example 4.2: In Example 3.9, $G_S(\mathbb{Z}_4 \oplus \mathbb{Z}_2)$ is bipartite by Corollary 4.1.

Definition 4.3: Let $I \neq 0$ be an ideal of R. By $G_S(R, I)$, we mean the subgraph of $G_S(R)$ whose vertices are I and all vertices adjacent to I along with the edges incident to these vertices. We call $G_S(R, I)$ the local semisimple-intersection graph of I.

Theorem 4.4: The graph $G_S(R)$ is bipartite if and only if $G_S(R, I)$ is a star graph with center I, for every nonzero simple ideal I of R.

***Proof*:** Assume $G_S(R)$ is a bipartite graph. Without lose of generality, assume $G_S(R)$ is not a null graph. Let V and W be a bipartition. Let I be a nonzero simple ideal of R. Suppose that $I \in V$. Then none the vertices of $N(I)$ belongs to V and therefore they belong to W. Thus, all vertices of $N(I)$ are pairwise nonadjacent. This means, that $G_S(R, I)$ is a star graph with center I. □

For the converse, assume that $G_S(R, I)$ is a star graph with center I, for every nonzero simple ideal $I \subseteq R$. Let V be the set of all nonzero simple ideals of R, and W be the set of the remaining (non-simple) ideals of R. By Remark 3.3, we have the vertices of V are not adjacent to each other. On the other hand, let $J, K \in W$. Assume, for contrary, that $J - K$. Then, $J \cap K$ is a nonzero simple ideal. Hence, $J, K \in V[G_S(R, J \cap K)] = N(J \cap K)$, which contradicts that $G_S(R, J \cap K)$ is a star graph with center $J \cap K$. Thus, we obtain that any two vertices of W are not adjacent. Consequently, $G_S(R)$ is bipartite with bipartition V and W. □

In General, a bipartite graph may have more than one bipartition. However, the existence of the (V,W)-bipartition of $G_S(R)$, where V is the set of all nonzero simple ideals of R, and W is the set of all non-simple ideals of R, is necessary and sufficient for $G_S(R)$ to be bipartite. We prove this in the next corollary, but before we do that we need to draw the attention of the reader to the fact that if an ideal does not include any simple ideal, then it is an isolated vertex which can be freely added to any part of a bipartition of $G_S(R)$. So, what maters when considering by partitions is the nonzero simple ideals and ideals containing anyone of them.

Corollary 4.5: Assume $G_S(R)$ is not null. Then, $G_S(R)$ is bipartite if and only if the (V,W)-bipartition of $G_S(R)$ exists, where V is the set of all nonzero simple ideals of R, and W is the set of all non-simple ideals of R.

***Proof*:** Assume $G_S(R)$ is bipartite. Let X and Y be a bipartition. Since $G_S(R)$ is not null, then R contains at least one nonzero simple (or semisimple) ideal. Without loss of generality, assume $R \in Y$. Since R is adjacent to every nonzero simple and semisimple ideals as Remark 3.3 states, then all nonzero simple ideals belong to X. Thus, all non-semisimple ideals containing simple ideals must belong to Y. The isolated vertices are distributed randomly to X and Y. Now if we move all isolated vertices from X to Y, the new bipartition is the (V,W)-bipartition with V is X after deleting the isolated vertices and W is Y plus all the isolated vertices. The proof of the converse is quite obvious. □

Recall that an ideal I of a ring R is large (or essential) if $I \cap J \neq 0$, for every nonzero ideal J of R. In what follows, when we assume $G_S(R)$ is bipartite, we consider the (V, W)-bipartition.

Theorem 4.6: Suppose $G_S(R)$ is not null and bipartite. Now the next statements are equivalent:
(1) $G_S(R)$ is complete bipartite.
(2) The intersection of all non-simple ideals is $Soc(R)$.
(3) $Soc(R)$ is an essential ideal.
(4) R is Artinian and $\text{diam}(G_S(R)) \leq 2$.
(5) $G_S(R)$ is path connected and $\text{diam}(G_S(R)) \leq 2$.

***Proof*:** The proof is easy when R contains only one nonzero simple ideal because $G_S(R)$ is a star graph and then (1) to (5) hold directly. Therefore, we assume that $G_S(R)$ contains more than one nonzero simple ideal. By Corollary 4.5, there exists the (V,W)-bipartition consisting of the set V of all nonzero simple ideals and the set W consisting of all non-simple ideals.

1⇒ 2**:** Suppose $G_S(R)$ is complete bipartite. Then every non-simple ideal is adjacent to every simple ideal. Hence, every non-simple ideal contains the socle of R. Thus, $Soc(R)$ is included in the intersection of all non-simple ideals. Now, $Soc(R)$ is not simple and hence it includes the intersection of all non-simple ideals. At this point, we conclude that $Soc(R)$ is equal to the intersection of all non-simple ideals.

2 ⇒ 3**:** The proof is trivial.

3 ⇒ 4**:** Assume that $Soc(R)$ is an essential ideal of R. The proof that R is Artinian is quite easy and well known in the literature. We only need to show that $\text{diam}(G_S(R)) \leq 2$. By Lemma 3.11, the distance between any two nonzero simple (or semisimple) ideals is 2. Let $I \in V$ and $J \in W$. If $I \subsetneq J$, then $I - J$ and $d(I,J) = 1$. Assume now $I \nsubseteq J$. Since $Soc(R)$ is large, then $Soc(R) \cap J \neq 0$, which means that J contains a nonzero simple ideal $K \neq I$. Then $I - I \oplus K - J$ is a path between I and J of length 2. Next, assume I, $J \in W$. We have $I \cap J$ is not simple. Since $Soc(R)$ is essential, the ideal $I \cap J$ contains a nonzero simple ideal K. Now the path $I - K - J$ between I and J has length 2.

4⇒ 5**:** Apply Theorem 3.12.

5⇒1**:** Assume $G_S(R)$ is path connected such that $\text{diam}(G_S(R)) \leq 2$. Let $I \in V$ and $J \in W$. Then, there exists a path connecting I and J. By assumption, the length of the shortest path connecting I and J equals 1 or 2. However, if the length of this shortest path is 2, then we get a contradiction to the hypothesis that says "$G_S(R)$ is bipartite". So, the length of this shortest path is 1. That is, I is adjacent to J. This ends the proof. □

Definition 4.7: Let I be a nonzero semisimple ideal of R. We say that the nonzero ideals J and K are adjacent through I if $J \cap K = I$.

Proposition 4.8: Every clique of $G_S(R)$ contains at most one nonzero simple ideal.

***Proof*:** The proof is straightforward from Remark 3.3. □

It follows from Proposition 4.8 that there are two types of cliques in $G_S(R)$. The first type of cliques contains no simple ideals, while the second type of cliques contains exactly one nonzero simple ideal. The next example exhibits these types of cliques.

Example 4.9: Let $R = I \oplus J \oplus K$ be a semisimple ring. Then, the subgraph $I \oplus J - J \oplus K - I \oplus K - I \oplus J$ is a clique whose vertices are not simple ideals. However, the subgraph $I \oplus J - I - I \oplus K - I \oplus J$ is a clique with one simple nonzero ideal as Proposition 4.8 emphasizes.

In the next work, we are going to study each type of cliques in order to discover the clique number of $G_S(R)$.

Theorem 4.10: Let Ω be a clique containing one nonzero simple (or semisimple) ideal I. Then Ω consists, beside the vertex I, vertices in $N(I)$ that are adjacent to each other through I.

***Proof*:** Let J and K be two vertices in Ω. Then $J \cap K$ is nonzero simple. Since $I - J$ and $I - K$, we get J, $K \in N(I)$ and $I \subseteq J \cap K$. Thus, $I = J \cap K$. The converse is obvious. □

Corollary 4.11: Let Ω be a clique containing one nonzero simple ideal I. Then $|V[\Omega]| > 2$ if and only if $R < V[\Omega]$ and $V[\Omega]$ has at least two proper non-semisimple ideals (that are adjacent to I).

***Proof*:** Suppose $V[\Omega]$ contains at least 3 vertices. Since I must be among the vertices of Ω, by Theorem 4.10, the other vertices are non-simple ideals that are adjacent to each other through I. By Remark 3.3, neither of the non-simple vertices equals R. The reverse is trivial. □

Definition 4.12: [12] Let I be a nonzero simple ideal of R. Then, the largest clique of $G_S(R)$ containing I is named the maximal clique induced by I.

Let I be a nonzero simple ideal of R. Then the clique $I - R$ is always a maximal clique induced by I, which we call the trivial maximal clique induced by I. It is not difficult to see from Corollary 4.11 that if $|N(I)| = 1$, then the trivial maximal clique induced by I is the only maximal clique induced by I. However, if $|N(I)| > 1$, Then there is another maximal clique induced by I which consists, in addition to I, of all proper non-simple ideals in $N(I)$ that are adjacent to each other through I. We denote this non-trivial maximal clique by $\Omega(I)$. Notice that $|V[\Omega(I)]| \geq 2$. Note that if I is semisimple but not simple ideal, we have $I - R$ is always a not clique in $G_S(R)$.

Example 4.13: In $G_S(\mathbb{Z}_4)$, the maximal cliques induced by the ideal $2\mathbb{Z}_4$ are only the trivial maximal clique $2\mathbb{Z}_4 - \mathbb{Z}_4$.

Example 4.14: In Example 4.9, $\Omega(I)$ is $I - I \oplus J - I \oplus K - I$. Thus $|\Omega(I)| = 3$.

5 CONCLUSION

In this paper, we have defined and studied an undirected graph $G_S(R)$, the semisimple-intersection graph of a ring R where the vertex set all nonzero ideals of R and two distinct vertices I and J are adjacent if and only if $I \cap J$ is a nonzero semisimple ideal. We observed that the graph $G_S(R)$ is complete if and only if every proper ideal of R either simple or semisimple and every pair of ideals in R have non-zero intersection. We studied girth, diameter, clique number, and bipartite property of the graph $G_S(R)$.

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REFERENCES

[1] S. Akbari, R. Nikandish, Some results on the intersection graph of ideals of matrix algebras, Linear Multilinear A., 62 (2014), 195-206.

[2] A. H. Alwan, Maximal submodule graph of a module, J. Discrete Math. Sci. Cryptogr., 24(7), 2021, 1941-1949.
[3] A. H. Alwan, Maximal ideal graph of commutative semirings, Int. J. Nonlinear Anal. Appl., 12(1), 2021, 913-926.
[4] A. H. Alwan, Small intersection graph of subsemimodules of a semimodule. Commun. Combin., Cryptogr. & Computer Sci., 2022; 1, 15-22.
[5] I. Beck, Coloring of commutative rings, J. Algebra 116 (1988), 208-226.
[6] J. A. Bondy, U. S. R. Murty, Graph theory, Springer-Verlag, London, 2011.
[7] J. Bosak, The graphs of semigroups, in Theory of Graphs and its Applications, (Academic Press, New York, 1964), pp. 119-125.
[8] I. Chakrabarty, S. Ghosh, T. K. Mukherjee, M. K. Sen, Intersection graphs of ideals of rings, Discrete Math., 309 (2009), 5381-5392.
[9] I. Chakrabarty, J. V. Kureethara, A survey on the intersection graphs of ideals of rings, Commun. Combin. Optim., 7 (2022), 121-167.
[10] P. M. Cohn, Introduction to ring theory, British Library Cataloguing in Publication Data, Springer-Verlag, London Berlin Heidelberg, 2000.
[11] J. Matczuk, M. Nowakowska, E. R Puczy lowski, Intersection graphs of modules and rings, J. Algebra Appl., 17 (2018), 185-131.
[12] F. Moh'd, M. Ahmed, Simple-intersection graphs of rings, AIMS Math., 8(1), 2022, 1040-1054.
[13] E. A. Osba, The intersection graph for finite commutative principal ideal rings, Acta Math. Acad. Paedagog. Nyházi., 32 (2016), 15-22.
[14] N. J. Rad, S. H. Jafari, S. Ghosh, On the intersection graphs of ideals of direct product of rings, J. Discuss. Math.-Gen. Algebra Appl., 34 (2014), 191-201.