Fuzzy e-Cosmall Submodules

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ABSTRACT

Let M be a module, μ and ν fuzzy submodules of M with $\mu \subseteq \nu$. Then μ is called a fuzzy e-cosmall submodule of ν in M if $\nu/\mu \ll_{fe} M/\mu\,(=\chi_M/\mu^*)$. In this paper we investigate fuzzy e-cosmall submodules. We also give some properties of fuzzy e-hollow modules.

KEYWORDS: Fuzzy small submodules, Fuzzy essential-small submodules, Fuzzy cosmall submodules, Fuzzy e-cosmall submodules, Fuzzy e-hollow modules.

1 INTRODUCTION

After the introduction of fuzzy sets by Zadeh in 1965 [9], a number of applications of this fundamental notion have come up. Naegoita and Ralescu [4] applied this notion to modules and defined fuzzy submodules of a module. So, fuzzy finitely generated submodules, fuzzy quotient modules [5], radical of fuzzy submodules, and primary fuzzy submodules [7, 2] were investigated. Kalita [12] defined a fuzzy essential submodule and proved some characteristics of such submodules. In 2011, Rahman and Saikia studied the concepts of fuzzy small submodules and Fuzzy cosmall submodules, in [6]. By using this idea, in [10] the authors investigated the Fuzzy cosmall submodules. Nimbhorkar and Khubchandani in 2020 [11] defined fuzzy essential-small (e-small) submodules. In this paper, we defined and studied fuzzy e-small submodules, fuzzy e-hollow modules and fuzzy e-cosmall submodules of a module. Throughout this paper R will denote an arbitrary associative ring with identity and M will be unital right R-module.

2 PRELIMINARIES

In this section, we briefly introduce some definitions and results of fuzzy sets and fuzzy submodules, which we need to develop our paper. By a fuzzy set of a module M we mean any mapping μ from M to [0, 1]. The support of a fuzzy set μ, denoted by μ^*, is a subset of M defined by $\mu^* = \{x \in M \mid \mu(x) > 0\}$. The subset μ_* of M is defined as $\mu_* = \{x \in M \mid \mu(x) = 1\}$. We denote the set of all fuzzy submodules of M by $F(M)$.

Definition 2.1. [3] Let M be an R-module. A fuzzy subset μ of M is said to be a fuzzy submodule, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:

(i) $\mu(0) = 1$;
(ii) $\mu(x - y) \geq min\{\mu(x), \mu(y)\}$;
(iii) $\mu(rx) \geq \mu(x)$.

Definition 2.2. [3] Let $\mu, \nu \in F(M)$ be such that $\mu \subseteq \nu$. Then the quotient of ν with respect to μ, is a fuzzy submodule of M/μ^* , denoted by ν/μ, and is defined as follows:

$$(\nu/\mu)([x]) = sup\{\nu(z) \mid z \in [x]\}, \forall x \in \nu^*$$

where $[x]$ denotes the coset $x + \mu^*$.

Lemma 2.3. [3] μ_* is a submodule of M if and only if μ is a fuzzy submodule of M. Let $N \leq M$, then characteristic function of N (χ_N) is defined as

$$\chi_N(x) = \begin{cases} 1, & \text{if } x \in N \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4. [3] Let $\mu \in F(M)$. Then $\mu_* = M$ if and only if $\mu = \chi_M$. Also if $\sigma \in F(M)$ and $\mu \subseteq \sigma$, then $\mu_* \subseteq \sigma_*$.

Lemma 2.5. [1] Let $\mu, \sigma \in F(M)$, then $(\mu \cap \sigma)_* = \mu_* \cap \sigma_*$, $(\mu \cup \sigma)_* = \mu_* \cup \sigma_*$. Further if μ and σ have finite images then $(\mu + \sigma)_* = \mu_* + \sigma_*$, where the sum of two fuzzy submodules is defined as $(\mu + \sigma)(x) = sup\{min\{\mu(a), \sigma(b)\} \mid a, b \in M, x = a + b\}$.

Definition 2.6. [3] Let $\mu, \sigma \in F(M)$. The sum $\mu + \sigma$ is called the direct sum of μ and σ if $\mu \cap \sigma = \chi_0$ and it is denoted by $\mu \oplus \sigma$.

Definition 2.7. [1] A fuzzy submodule $\mu(\neq \chi_0)$ of M is said to be fuzzy indecomposable if there is not $(\chi_0, \chi_M \neq)\sigma, \nu \in F(M)$ such that $\mu = \nu \oplus \sigma$.

Definition 2.8. [1] Let M be an R-module and $\mu \in F(M)$. Then μ is called a fuzzy small submodule of M if for any $\nu \in F(M)$, $\mu + \nu = \chi_M$ implies that $\nu = \chi_M$. It is indicated by the notation $\mu \ll_f M$ or $\mu \ll_f \chi_M$. Equivalently, if for any $\nu \in F(M)$ satisfying $\nu \neq \chi_M$ implies $\mu + \nu \neq \chi_M$.

It is obvious that χ_0 is always a fuzzy small submodule of M.

A submodule N of M is named small in M, symbolized by $N \ll M$, if $M \neq N + K$ for any proper submodule K of M [8].

Lemma 2.9. [1] Let $\mu \in F(M)$. Then $\mu \ll_f M$ if and only if $\mu_* \ll M$.

Recall that [8] a submodule N of an R-module M is called an essential submodule of M, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$.

Definition 2.10. [11] Let M be an R-module and $\mu \in F(M)$. Then μ is called a fuzzy essential submodule of M, if for any $\nu \in F(M)$ satisfying $\mu \cap \nu = \chi_\theta$ implies $\nu = \chi_\theta$ and is denoted by $\mu \trianglelefteq_f M$.

Theorem 2.11. [12] A submodule N of an R-module M is essential in M if and only if χ_N is an essential fuzzy submodule of M.

Theorem 2.12. [12] Let μ be a non-zero fuzzy submodule of an R-module M. Then $\mu \trianglelefteq_f M$ if and only if $\mu^* \trianglelefteq M$.

3 FUZZY *E*-HOLLOW MODULES AND FUZZY *E*-COSMALL SUBMODULES

In this section, we introduce the concept of a fuzzy essential-cosmall (for simply e-cosmall) submodule. We obtain some properties of this concept. Now onwards all the fuzzy sets involved in this paper have finite images.

Let μ and σ be any two fuzzy submodule of M such that $\mu \subseteq \sigma$, then μ is called a fuzzy submodule of σ. μ is called a fuzzy small submodule in σ, denoted by $\mu \ll_f \sigma$, if $\mu \ll_f \sigma^*$.

Definition 3.1. [15] Let N be a submodule of an R-module M. N is said to be e-small in M (denoted by $N \ll_e M$), if $N + L = M$ with $L \trianglelefteq M$ implies $L = M$.

For more information about e-small submodules we refer to [14].

Definition 3.2. [11] Let M be an R-module and let $\mu \in F(M)$. Then μ is called a fuzzy e-small (essential-small or generalized small) submodule of M if for any essential submodule $\sigma \in F(M)$, $\mu + \sigma = \chi_M$ implies that $\sigma = \chi_M$.

Theorem 3.3. [11] Let $\mu \in F(M)$. Then $\mu \ll_{fe} M$ if and only if $\mu_* \ll_e M$.

We note that every fuzzy small submodule of an R-module is a fuzzy e-small submodule. However, the following example shows that the converse need not be true.

Example 3.4. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{24}$.

Define $\mu: M \longrightarrow [0,1]$ by,

$$\mu(x) = \begin{cases} 1, & \text{if } x \in \{0,8,16\}, \\ \alpha, & \text{if } x \notin \{0,8,16\}, \text{where } 0 \leq \alpha < 1. \end{cases}$$

We note that $\mu_* = \{0, 8, 16\}$ is an e-small submodule of $\mathbb{Z}_{24}$. Hence, μ is a fuzzy e-small submodule of $\mathbb{Z}_{24}$ by Theorem 3.3. But, $\mu_* = \{0, 8, 16\}$ is not a small submodule of $\mathbb{Z}_{24}$ and so by Lemma 2.9, μ is not a fuzzy small submodule of $\mathbb{Z}_{24}$.

Recall that [13] the module M is called an e-hollow (or generalized hollow) module if every proper submodule of M is e-small (or generalized small) in M. In this case, it is clear that every hollow module is an e-hollow module.

Definition 3.5. [10] Let M be an R-module. χ_M is said to be a fuzzy hollow module, if every proper fuzzy submodule of χ_M is fuzzy small submodule in χ_M.

Similar to Definition 3.5, we have the following definition.

Definition 3.6. Let M be an R-module. χ_M is said to be a fuzzy e-hollow module, if every proper fuzzy submodule of χ_M is fuzzy e-small submodule in χ_M or M has no proper fuzzy essential submodules.

In this case, it is clear that every fuzzy hollow module is a fuzzy e-hollow module.

Similar to [10, Proposition 3.2], we have the following proposition.

Proposition 3.7. Let M be a module. Then:
(i) If χ_M is a fuzzy e-hollow module, then every factor of χ_M is fuzzy e-hollow.
(ii) If χ_M is fuzzy e-hollow, then χ_M is fuzzy indecomposable.
(iii) If $\sigma \ll_{fe} \chi_M$ and χ_M/σ is fuzzy e-hollow, then χ_M is fuzzy e-hollow.

Proof. (i) By using [1, Proposition 3.8].
(ii) It is clear.
(iii) Let $\mu \in F(M)$ be a proper fuzzy submodule of χ_M. Assume that $\mu + \nu = \chi_M$ for ν is fuzzy essential submodule of χ_M. Then $\mu + \nu + \sigma = \chi_M$. Hence $(\mu + \sigma)/\sigma + (\nu + \sigma)/\sigma = \chi_M/\sigma$. Since χ_M/σ is fuzzy e-hollow and $(\nu + \sigma)/\sigma$ is fuzzy essential submodule of χ_M/σ, then $(\nu + \sigma)/\sigma = \chi_M/\sigma$. Thus $\nu + \sigma = \chi_M$. As $\sigma \ll_{fe} \chi_M$, $\nu = \chi_M$. So χ_M is fuzzy e-hollow. □

Let M be a module and $N \leq L \leq M$. Recall that N is called a cosmall submodule of L in M if $L/N \ll M/N$. The notation $N \overset{cs}{\underset{M}{\hookrightarrow}} L$ indicates that N is a cosmall submodule of L in M [8]. In [6], authors generalized this concept in fuzzy settings. N is called an e-cosmall submodule of L in M if $L/N \ll_e M/N$. The notation $N \overset{e-cs}{\underset{M}{\hookrightarrow}} L$ indicates that N is a e-cosmall submodule of L in M. In this section, we define fuzzy e-cosmall submodules.

Definition 3.8. [6] Let M be a module and Let $\mu, \nu \in F(M)$ with $\mu \subseteq \nu$. Then μ is called a fuzzy cosmall submodule of ν in M if $\nu/\mu \ll_f M/\mu$ $(= \chi_M/\mu^*)$. The notation $\mu \overset{fcs}{\underset{M}{\hookrightarrow}} \nu$ indicates that μ is a fuzzy cosmall submodule of ν in M.

Definition 3.9. Let M be a module and Let $\mu, \nu \in F(M)$ with $\mu \subseteq \nu$. Then μ is called a fuzzy e-cosmall submodule of ν in M if $\nu/\mu \ll_{fe} M/\mu$ $(= \chi_M/\mu^*)$. The notation $\mu \overset{fe-cs}{\underset{M}{\hookrightarrow}} \nu$ indicates that μ is a fuzzy e-cosmall submodule of ν in M.

Example 3.10. Consider $M = \mathbb{Z}_8 = \{0,1,2,3,4,5,6,7\}$ under addition modulo 8. Then M is a module over the ring $\mathbb{Z}$. Let $S = \{0,2,4,6\}$. Define $\mu: M \to [0,1]$ as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x \in S \\ \alpha & \text{otherwise} \end{cases}$$

where $0 \leq \alpha < 1$. Then χ_0 is a fuzzy e-cosmall submodule of μ in M.

Similar to [10, Proposition 3.5], we have the following proposition.

Proposition 3.11. Let $\mu, \sigma \in F(M)$. Then $\mu \overset{fe-cs}{\underset{M}{\hookrightarrow}} \sigma$ if and only if $\mu_* \overset{e-cs}{\underset{M}{\hookrightarrow}} \sigma_*$.

Proof. Let $\mu, \sigma \in F(M)$. $\mu \overset{fe-cs}{\underset{M}{\hookrightarrow}} \sigma$ if and only if $\sigma/\mu \ll_{fe} M/\mu$. By Theorem 3.3, $(\sigma/\mu)_* \ll_e (M/\mu)_*$. Thus, $\sigma_*/\mu_* \ll_e M/\mu_*$. Hence, $\mu_* \overset{e-cs}{\underset{M}{\hookrightarrow}} \sigma_*$. □

Proposition 3.12. Let M be a module. If $\mu \subseteq \sigma \subseteq \chi_M$ and $\sigma = \mu + \nu$, where $\nu \ll_{fe} \mu$, Then μ is a fuzzy e-cosmall submodule of σ in M.

Proof. Let $\chi_M = \sigma + \gamma$ for some $\gamma \in F(M)$. Then $\chi_M = \mu + \nu + \gamma = \mu + \gamma$, since $\nu \ll_{fe} M$. Thus, by [6, Theorem 4.24], μ is a fuzzy cosmall submodule of σ in M. Hence, μ is a fuzzy e-cosmall submodule of σ in M. □

Definition 3.13. [3] Let X and Y be any two nonempty sets, and $f: X \to Y$ be a mapping. Let μ be a fuzzy subset of X and σ a fuzzy subset of Y, then the image $f(\mu)$ and the inverse image $f^{-1}(\sigma)$ are defined as follows: for all $y \in Y$

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) | x \in X, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

And $f^{-1}(\sigma)(x) = \sigma(f(x))$ for all $x \in X$.

Lemma 3.14. Let $f: M \to N$ be an epimorphism. If $\mu \in F(M)$ and $\sigma \in F(N)$, then:

(iv) $f(\mu)_* = f(\mu_*)$;
(v) $f^{-1}(\sigma)_* = f^{-1}(\sigma_*)$.

Proof. (i) Let $x \in f(\mu)_*$, then $(f(\mu))(x) = 1$. Thus $sup\{\mu(y) \mid f(y) = x\} = 1$. Thus $\mu(y) = 1$ for some $y \in f^{-1}(x)$. Then $y \in \mu_*$, where $f(y) = x$, that is, $x \in f(\mu_*)$. So $f(\mu)_* \subseteq f(\mu_*)$. Conversely, let $x \in f(\mu_*)$, then $x = f(y)$, for some $y \in \mu_*$. So $y \in f^{-1}(x)$, where $\mu(y) = 1$. Thus $sup\{\mu(y) \mid y \in f^{-1}(x)\} \geq \mu(y) = 1$. Hence $(f(\mu))(x) = 1$, and so $x \in f(\mu)_*$. Therefore, the equality follows.

(ii) $x \in f^{-1}(\sigma)_*$ if and only if $f^{-1}(\sigma)(x) = 1$ if and only if $\sigma(f(x)) = 1$ if and only if $f(x) \in \sigma_*$ if and only if $x \in f^{-1}(\sigma_*)$. □

Theorem 3.15. Let $f: M \to N$ be an epimorphism. If $\mu \subseteq \nu \subseteq \chi_M$ and $\mu \overset{fe-cs}{\underset{M}{\hookrightarrow}} \nu$, then $f(\mu) \overset{fe-cs}{\underset{N}{\hookrightarrow}} f(\nu)$.

Proof. Let $\sigma \subseteq \chi_N$ and $f(\nu) + \sigma = \chi_N$. Then, by Proposition 3.12 and Lemma 3.14, $f(\nu_*) + \sigma_* = N$. Since f is an epimorphism, there exists $K \leq M$ such that $f(K) = \sigma_*$. It is clear that $K = (\chi_K)_*$ and so $f(K) = f((\chi_K)_*) = \sigma_*$. Hence $f(\nu_* + (\chi_K)_*) = f(\nu_*) + f((\chi_K)_*) = N = f(M)$. So $\nu_* + (\chi_K)_* = M$. By Lemma 2.5 and Lemma 2.4, $\nu + \chi_K = \chi_M$. By hypothesis, $\mu + \chi_K = \chi_M$. Then $\mu_* + (\chi_K)_* = M$. Thus $f(\mu_*) + f((\chi_K)_*) = N$, so $f(\mu_*) + \sigma_* = N$. So $f(\mu) + \sigma = \chi_N$. By [6, Theorem 4.24], $f(\mu) \overset{fcs}{\underset{N}{\hookrightarrow}} f(\nu)$. Hence, $f(\mu) \overset{fe-cs}{\underset{N}{\hookrightarrow}} f(\nu)$. □

Theorem 3.16. Let $f: M \to N$ be an epimorphism. If $\nu \subseteq \sigma \subseteq \chi_N$, then $\nu \overset{fe-cs}{\underset{N}{\hookrightarrow}} \sigma$ if and only if $f^{-1}(\nu) \overset{fe-cs}{\underset{M}{\hookrightarrow}} f^{-1}(\sigma)$.

Proof. Let $\mu \in F(M)$ and $f^{-1}(\sigma) + \mu = \chi_M$. By Proposition 3.12, $(f^{-1}(\sigma))_* + \mu_* = M$. From Lemma 3.14, $f^{-1}(\sigma_*) + \mu_* = M$. Thus $\sigma_* + f(\mu_*) = N$. Using Proposition 3.11, $\nu_* \overset{e-cs}{\underset{N}{\hookrightarrow}} \sigma_*$, and so $\nu_* + f(\mu_*) = N$. Hence $f^{-1}(\nu_*) + \mu_* = M$. Then $f^{-1}(\nu) + \mu = \chi_M$. Therefore $f^{-1}(\nu) \overset{fcs}{\underset{M}{\hookrightarrow}} f^{-1}(\sigma)$, by [6, Theorem 4.24]. Thus, $f^{-1}(\nu) \overset{fe-cs}{\underset{M}{\hookrightarrow}} f^{-1}(\sigma)$.

Conversely, let $\sigma + \mu = \chi_N$ for some $\mu \in F(M)$. Then $\sigma_* + \mu_* = N$, so $f^{-1}(\sigma_*) + f^{-1}(\mu_*) = M$. Thus $f^{-1}(\sigma) + f^{-1}(\mu) = \chi_M$. Hence $f^{-1}(\nu_*) + f^{-1}(\mu_*) = M$, then $\nu_* + \mu_* = N$. So $\nu + \mu = \chi_N$. Therefore $\nu \overset{fcs}{\underset{N}{\hookrightarrow}} \sigma$. Hence, $\nu \overset{fe-cs}{\underset{N}{\hookrightarrow}} \sigma$. □

4 CONCLUSION

In this paper, we have defined and studied a fuzzy e-cosmall submodules. We observed that if $f: M \to N$ be an epimorphism with $\mu \subseteq \nu \subseteq \chi_M$ and $\mu \overset{fe-cs}{\underset{M}{\hookrightarrow}} \nu$, then $f(\mu) \overset{fe-cs}{\underset{N}{\hookrightarrow}} f(\nu)$. We also studied fuzzy e-hollow modules.

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