Two upper bounds for the number of spanning trees in a square lattice

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Abstract

For a connected graph G, we denote the number of spanning trees in this graph by $\tau(G)$. Although, the Kirchhoff's Matrix Tree theorem provides a relation for the exact solution of this problem, but computing this exact solution requires finding the Laplacian determinant, which is complicated and time-consuming when the number of vertices increases. Therefore, presenting an upper bound is important for this problem. In this paper, we study the problem of counting the number of spanning trees in the square lattice graph G_n with n^2 vertices, and present two upper bounds $\tau(G_n) < 2^4 \times 3^{4(n-2)} \times 4^{(n-2)^2}$ and $\tau(G_n) < \prod_{i=1}^{n} \binom{4(n-i)}{\frac{4(n-i)}{2}}$ for the number of spanning trees in this graphs.

Keywords: Spanning trees, Square lattice, Grid graphs.

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1 Introduction

Counting spanning trees in a graph or lattice is one of the long-standing and favorite problems in physics and mathematics [1, 2, 3]. Let $G = (V, E)$ be a simple connected graph with a set of vertices V and a set of edges E, so that $n = |V|$ is the number of vertices and $e = |E|$ is the number of edges of the graph. A spanning tree $T(V, E')$ is a subgraph of $G(V, E)$ such that it contains all the vertices of G and its edge set is $E' \subseteq E$, but has no cycle and $e = n - 1$. We denote the number of spanning trees in a graph G by $\tau(G)$. In this paper, we investigate the problem of counting the number of spanning trees in the square lattice graphs. A square lattice G_n is an $n \times n$ lattice graph with the set of vertices $V = \{1, \cdots, n\} \times \{1, \cdots, n\}$ and both vertices (i, j) and (i', j') that are adjacent if $|i - i'| + |j - j'| = 1$. Square lattices are important in statistical physics [4].

One of the fascinating results presented by a German physicist named Gustav Kirchhoff in the middle of the 19th century is the Kirchhoff's Matrix Tree theorem, which could calculate the number of spanning trees in a graph.

Theorem 1.1 (Kirchhoff′s Matrix Tree Theorem [5])**.** *Let $G(V, E)$ be an undirected graph with n vertices, then for the Laplacian graph $L = \{l_{ij}\}$, the number of spanning trees in G can be calculated as follows*

$$\tau(G) = det(L'(j))$$

[1]speaker

where $L'(j)$ is a matrix that is obtained by choosing a vertex v_j and removing the jth row and column from the matrix L and

$$l_{ij} = \begin{cases} deg(v_i) & if \quad i = j \\ -1 & if \quad i \neq j \quad and \quad v_i \quad adjacent \quad to \quad v_j \\ 0 & Otherwise \end{cases}$$

Although, the Kirchhoff's Matrix Tree theorem provides a relation for calculating the number of spanning trees in a graph, but with the increasing the number of vertices and the size of the Laplacian determinant, these calculations become complicated and time-consuming. Therefore, in many cases, we seek to provide an upper bound for this problem [6, 7, 8, 9]. In this paper, we present two upper bounds for the number of spanning trees in a square lattice in terms of the number of vertices.

2 Main results

In this section, we present two upper bounds for the number of spanning trees in a square lattice. We have the first bound as follows.

Theorem 2.1. *Let G_n be a square lattice graph with n^2 vertices, then we have the following upper bound for the number of spanning trees in the square lattice graph G_n:*

$$\tau(G_n) < 2^4 \times 3^{4(n-2)} \times 4^{(n-2)^2}.$$

Proof. Since a spanning tree must contain all vertices of G_n, suppose that we number the vertices of a square lattice from 1 to n^2 and then start with vertex number 1 and choose one of the vertices connected to it, and repeat this process for all vertices. According to this process, because a square lattice has 4 vertices of degree 2, $4(n-2)$ vertices of degree 3 and $(n-2)^2$ vertices of degree 4, we have $2^4 \times 3^{4(n-2)} \times 4^{(n-2)^2}$ choices to perform this process. Therefore, the number of spanning trees in a square lattice graph cannot be greater than this value, and therefore this value is an upper bound for this problem. $\square$

Now, we want to give another upper bound for this problem using another method.

Theorem 2.2. *For a square lattice graph G_n with n^2 vertices, we have the following upper bound for the number of spanning trees in G_n:*

$$\tau(G_n) < \prod_{i=1}^{n} \binom{4(n-i)}{\frac{4(n-i)}{2}}.$$

Proof. Suppose, we want to calculate the number of spanning trees in a square lattice graph G_n with n^2 vertices. For this purpose, consider two square lattice graphs G_n, G_{n-1} and their difference as shown in Figure 1. Consider we have a spanning tree T_{n-1} in a square lattice graph G_{n-1} with $(n-1)^2$ vertices.

Now, suppose that according to Figure 2, we want to add new edges and vertices to the spanning tree T_{n-1} to find a spanning tree T_n of the square lattice graph G_n. It is clear that no edge from the square lattice graph G_{n-1} cannot be added to the spanning tree T_{n-1}, because it creates a cycle. Therefore, the edges must be from the edges set $E(G_n - G_{n-1})$. Since, for a square lattice graph G_n, the number of vertices is n^2 and the number of edges is $2n(n-1)$, so, the number of these reminded edges is:

$$|E(G_n)| - |E(G_{n-1})| = 2n(n-1) - 2(n-2)(n-1)$$

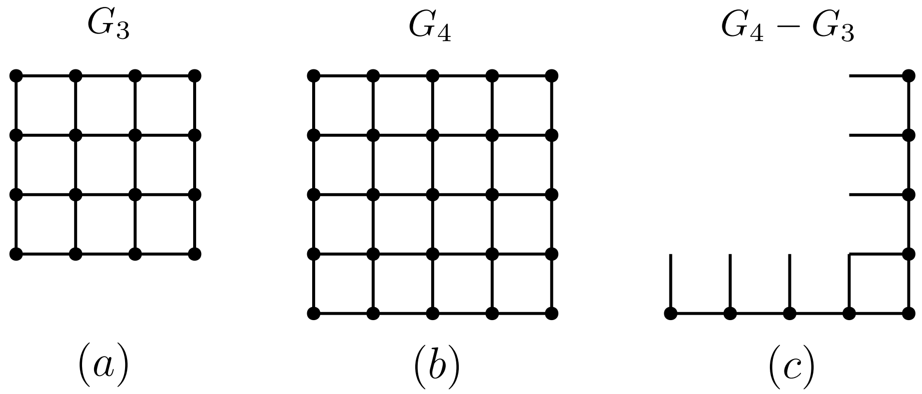

Figure 1: Two square lattice graphs G_3 and G_4 and the set of edges related to the difference of these two graphs.

$$
\begin{aligned}
&= 2n^2 - 2n - 2(n^2 - 2n - n + 2) \\
&= 4n - 4 \\
&= 4(n-1).
\end{aligned}
$$

On the other hand, in order to get from spanning tree T_{n-1} to the spanning tree T_n, which is a spanning tree of G_n, we need to select $\frac{4(n-1)}{2} + 1$ edges from the $4(n-1)$ remained edges. See Figure 2.

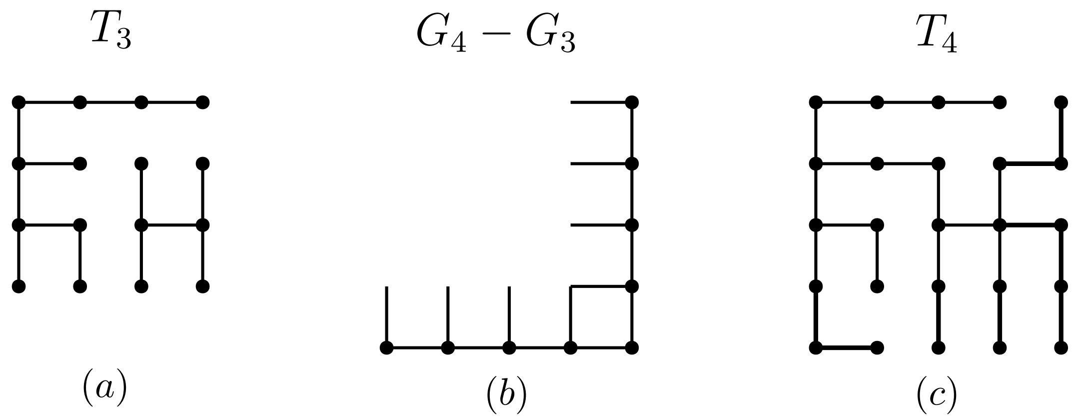

Figure 2: Obtaining spanning tree T_4 from the spanning tree T_3 by selecting the number of edges from the graph $G_4 - G_3$.

Therefore, we will have the following relation between the number of spanning trees of the square lattice graphs G_n and G_{n-1}:

$$
\begin{aligned}
\tau(G_n) &< \tau(G_{n-1}) \times \binom{4(n-1)}{\frac{4(n-1)}{2}+1} \\
&< \tau(G_{n-1}) \times \binom{4(n-1)}{\frac{4(n-1)}{2}} \\
&< \tau(G_{n-2}) \times \binom{4(n-2)}{\frac{4(n-2)}{2}} \times \binom{4(n-1)}{\frac{4(n-1)}{2}} \\
&\quad \vdots \\
&< \tau(G_1) \times \binom{4(2-1)}{\frac{4(2-1)}{2}} \times \binom{4(3-1)}{\frac{4(3-1)}{2}} \times \quad \cdots \quad \times \binom{4(n-2)}{\frac{4(n-2)}{2}} \times \binom{4(n-1)}{\frac{4(n-1)}{2}} \\
&< 1 \times 6 \times 70 \times \quad \cdots \quad \times \binom{4(n-2)}{\frac{4(n-2)}{2}} \times \binom{4(n-1)}{\frac{4(n-1)}{2}}
\end{aligned}
$$

$$= \prod_{i=1}^{n} \binom{4(n-i)}{\frac{4(n-i)}{2}}.$$

and this completes the proof. □

3 Conclusions

In this paper, we study the problem of computing the number of spanning trees in a square lattice and present two upper bounds for this problem. Further work can be about improving these upper bounds and generalizing them for special cases of the grid graphs.

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