8th International Conference on
Combinatorics, Cryptography, Computer Science and Computation
November. 15-16, 2023

Accessible Binary Code Based On Q-algebras

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Abstract

In this article, we define the notion of a Q-algebra-valued function on a set and investigate related properties. We establish the binary block codes generated by Q-algebras valued function to unique up to isomorphism. In this study the connection between Q-algebras and any given finite set via the logic algebra-valued functions and valued-cuts play an essential role in constructing binary codes.

Keywords: Q-algebra, Q-valued function, Q-binary code, valued-cut.

AMS Mathematical Subject Classification [2010]: 03B70 , 13-04

1 Introduction

In computer science, a block code is a type of channel coding. It adds redundancy to a message so that, at the receiver, one can decode the message with a minimum number of errors, where it is already provided that the information rate would not exceed the channel capacity. Block codes can be source codes used in data compression, or channel codes used for detection and correction of channel errors [2]. A set alone cannot have practical use, but when it becomes a universal algebra and is equipped with several operations, it becomes a system and finds basic applications. One of the most important algebras in universal algebra is logical algebras. The theory of logical algebra is one of the most important branches of mathematics, which is widely used in interdisciplinary sciences. For instance, computer engineering sciences and computer calculations are formed based on two-valued logic, therefore logical algebras that are made based on binary operations are used and have special importance in computer engineering sciences. Neggers et al. introduced a new notation of Q-algebra as a generalization of the idea of BCI-algebras and generalized some theorems discussed in BCI- algebras [3]. Block codes in different logical algebras have been studied by numerous researchers in the past few years. Jun and Song [1] introduced codes based on BCK-algebras whereas Surdive et al. studied coding theory in hyper BCK-algebras [4]. The objective of this paper is to introduce the concept of block code based on the Q-algebra. We try to extract the binary codes from any given Q-algebra and any given finite set via the finite Q-valued functions and cut functions.

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Definition 1.1. [3] A Q-algebra is a nonempty set X with a constant 0_X and a binary operation $*$ satisfying axioms:

(i) $x * x = 0_X$,

(ii) $x * 0_X = x$,

(iii) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

In $(X, *, 0_X)$, we can define a binary relation $\leq_X$ by $x \leq_X y$ if and only if $x * y = 0_X$.

Theorem 1.2. [3] *Let $(X, *)$ be a Q-algebra and $x, y, z \in X$. Then*

(i) *If for all $x \in X, x \leq_X 0$, then X contains only* 0.

(ii) *If $x \leq_X y$, then $x * (x * (x * y)) = 0$.*

(iii) *If $x \leq_X y$ and $x * z \leq_X w$, then $0 \leq_X z$.*

2 On Q-valued functions

In this section, introduce the concept of Q-valued functions and cut functions and investigate the properties of Q-valued functions based on valued-cut.

Definition 2.1. Let $(X, *, 0_X)$ be a Q-algebra, $\emptyset \neq Y$ be an arbitrary set and $w \in X$. Then

(i) a map $\varphi : Y \to X$ is called a Q-valued function on Y.

(ii) a map $\varphi^w : Y \to \{0, 1\}$ is called a cut function on Y, if for any $x \in Y, \varphi^w(x) = \begin{cases} 1 & \text{if } \varphi(x) * w = 0_X \\ 0 & \text{if } \varphi(x) * w \neq 0_X \end{cases}$

and $\varphi^{(w)} = \{y \in Y \mid \varphi(y) * w = 0_X\}$ is called an w-cut of φ.

Example 2.2. Let $X = \{0_X, a, b, c\}$ and $Y = \{m, p, q\}$. Then $(X, *)$ is a Q-algebra as Table 2.

$*$	0_X	a	b	c
0_X	0_X	a	b	c
a	a	0_X	c	b
b	b	c	0_X	a
c	c	b	a	0_X

Table 1: Q-algebra $(X, *, 0_X)$

Define $\varphi : Y \to X$, by $\varphi = \begin{pmatrix} m & n & p \\ a & b & c \end{pmatrix}$, then

$$\varphi^{0_X} = \begin{pmatrix} m & n & p \\ 0 & 0 & 0 \end{pmatrix}, \varphi^a = \begin{pmatrix} m & n & p \\ 1 & 0 & 0 \end{pmatrix}, \varphi^b = \begin{pmatrix} m & n & p \\ 0 & 1 & 0 \end{pmatrix}, \varphi^c = \begin{pmatrix} m & n & p \\ 0 & 0 & 1 \end{pmatrix}, \varphi^{(0_X)} = \emptyset,$$
$\varphi^{(a)} = \{m\}, \varphi^{(b)} = \{n\}$ *and* $\varphi^{(c)} = \{p\}$.

Theorem 2.3. *Let $(X, *, 0_X)$ be a Q-algebra and $\varphi : Y \to X$ be a Q-valued function. Then*

(i) *if φ is an onto, then for any $w \in X, \varphi^{(w)} \neq \emptyset$,*

(ii) *if φ is an onto, then for any* $\bigcup_{w\in X} \varphi^{(w)} = Y$,

(iii) *if $(X,*)$ is a semigroup and $w' * w = 0_X$, then $\varphi^{(w')} \subseteq \varphi^{(w)}$, which $w, w' \in X$.*

Let $(X,*,0_X)$ be a Q-algebra and $\emptyset \neq A \subseteq X$. Denote $C(A) = \{a \in A \mid 0_X * a = a\}$ as stabilizer of A, $N(A) = \{a \in A \mid 0_X * a = 0_X\}$ as constanter of A and for any $w \in X, U(w) = \{x \in X \mid x * w = 0_X\}$ as the set of left units of w. Thus have the following results.

Corollary 2.4. *Let $(X,*,0_X)$ be a Q-algebra and $\varphi : Y \to X$ be a Q-valued function. Then*

(i) *for any $y \in Y, \varphi^{(\varphi(y))} \neq \emptyset$,*

(ii) $\bigcup_{y\in Y} \varphi^{(\varphi(y))} = Y$.

Theorem 2.5. *Let $(X,*,0_X)$ be a Q-algebra, $\varphi : Y \to X$ be a Q-valued function and $C(X) = X$. Then*

(i) *$(X, \leq_X)$ is a poset.*

(ii) *for any $x \in X, \varphi(x) = \bigwedge\{w \in X \mid \varphi(x) * w = 0_X\}$.*

(iii) *if $w * w' = 0_X$, then $\varphi^{(w')} = \varphi^{(w)}$, which $w, w' \in X$*

Theorem 2.6. *Let $(X,*,0_X)$ be a Q-algebra, $\varphi : Y \to X$ be a Q-valued function and $C(X) = X$. Then*

(i) *for any $x, y \in Y, \varphi(x) \neq \varphi(y)$ if and only if $\varphi^{(\varphi(x))} \neq \varphi^{(\varphi(y))}$.*

(ii) *for any $w \in X$ and any $y \in Y, \varphi(y) * w = 0_X$ if and only if $\varphi^{(\varphi(y))} \subseteq \varphi^{w}$.*

(iii) *for any $x, y \in Y, \varphi(x) * \varphi(y) = 0_X$ if and only if $\varphi^{(\varphi(x))} \subseteq \varphi^{(\varphi(y))}$.*

Let $(X,*,0_X)$ be a Q-algebra, $\varphi : Y \to X$ be a Q-valued function. Then define a binary operation R on X defined by $(w, w') \in R$ if and only if $\varphi^{(w)} = \varphi^{(w')}$. It is easy to see that R is an equivalence relation on X.

Theorem 2.7. *Let $(X,*,0_X)$ be a Q-algebra and $\varphi : Y \to X$ be a Q-valued function. Then for any $w, w' \in X$:*

(i) *$N(w) = N(w')$ if and only if $0 * w = 0 * w'$.*

(ii) *$R(w) = R(w')$ if and only if $N(w) \cap \varphi(Y) = N(w') \cap \varphi(Y)$.*

Corollary 2.8. *Let $(X,*,0_X)$ be a Q-algebra, $\varphi : Y \to X$ be a Q-valued function and $w, w' \in X$.*

(i) *If $C(X) = X$, then $N(w) = N(w')$ if and only if $w = w'$.*

(ii) *If φ is surjective, then $R(w) = R(w')$ if and only if $N(w) = N(w')$.*

(iii) *If $C(X) = X$ and φ is surjective, then $R(w) = R(w')$ if and only if $w = w'$.*

Let $(X,*,0_X)$ be a Q-algebra and $\varphi : Y \to X$ be a Q-valued function. Then consider $\varphi^X = \{\varphi^w \mid w \in X\}$ and $\varphi^{(X)} = \{\varphi^{(w)} \mid w \in X\}$. Then have the following results.

Theorem 2.9. *Let $(X, *, 0_X)$ be a Q-algebra and $\varphi : Y \to X$ be a Q-valued function, which $(X, \leq_Y)$ is a poset. Then*

(*i*) *for all* $A \subseteq X, \varphi^{(Inf(A))} = \bigcap_{a \in A} \varphi^{(a)}$.

(*ii*) $Y = \bigcup_{a \in X} \varphi^{(a)}$.

3 Codewords generated by Q-valued functions

In this section, we establish codewords in a binary block-code generated by a Q-valued function. Finally, we prove that every finite Q-algebra which has the order less than or equal to the order of a finite set determines a binary block-code which is isomorphic to it.

Definition 3.1. Let $(X, *, 0_X)$ be a Q-algebra, $Y = \{1, 2, 3, \dots, n\}$ and $\varphi : Y \to X$ be a Q-valued function. For any $n \in \mathbb{N}$ and any $w \in X$, correspond the class $R(w)$ to a codeword $v_w = w_1 w_2 w_3 \dots w_n$, which for any $1 \leq i \leq n, w_i = \begin{cases} 1 & \text{if } \varphi^w(i) = 1 \\ 0 & \text{if } \varphi^w(i) = 0 \end{cases}$. The binary block-code V of length n, will denote by $V = \{v_w \mid w \in X\}$.

Theorem 3.2. *Let $(X, *, 0_X)$ be a Q-algebra, $w, w' \in X$. Then $R(w) = R(w')$ if and only if $v_w = v_{w'}$.*

Proof. Let $w, w' \in X$. Then $R(w) = R(w')$ if and only if $\varphi^{(w)} = \varphi^{(w')}$. Then $R(w) = R(w') \Leftrightarrow \{i \in Y \mid \varphi^w(i) = 1\} = \{j \in Y \mid \varphi^{w'}(j) = 1\} \Leftrightarrow w_i = w'_i \Leftrightarrow v_w = v_{w'}$. □

Let $v_w = w_1 w_2 w_3 \dots w_n$ and $v'_w = w'_1 w'_2 w'_3 \dots w'_n$ be two codewords belonging to a binary block-code V. Define an order relation $\leq_V$ on V by $v_w \leq_V v'_w$ if and only if for all $i \in Y, w_i \leq w'_i$. It is clear that $(V, \leq_V)$ is a poset.

Theorem 3.3. *Let $(X, *, 0_X)$ be a Q-algebra, $w, w' \in X$. Then $v_w \leq_V v'_w$ if and only if $\varphi^{(w)} \subseteq \varphi^{(w')}$.*

Proof. Let $w, w' \in X$ and $\varphi^{(w)} \subseteq \varphi^{(w')}$. Then for any $x \in \varphi^{(w)}, w = \varphi(x) = w'$ and so $w = w'$. Thus, $v_w \leq_V v'_w$. □

Example 3.4. (*i*) Let $Y = \{1, 2, 3, 4\}$. Consider the Q-algebra $(C, *, 0_X)$ in Figure 2 and $(X, \leq_X)$ in Figure 1. Define $\varphi : Y \to X$, by $\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & 0_X \end{pmatrix}$, then

$$\varphi^{0_X} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \varphi^a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \varphi^b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \varphi^c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \varphi^{(0_X)} = \{4\},$$
$\varphi^{(a)} = \{1\}, \varphi^{(b)} = \{2\}$ *and* $\varphi^{(c)} = \{3\}$.

Then the equivalence relation R on X is as $R = \{(0, 0), (a, a), (b, b), (c, c)\}$. Hence, we have all distinct codewords of the binary block-code $V = \{v_{0_X} = 0001, v_a = 1000, v_b = 0100, v_c = 0010\}$ and so have the poset $(V, \leq_V)$ in Figure 2.

Figure 1: $(X, \leq_X)$

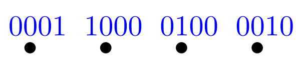

Figure 2: $(V, \leq_V)$

(ii) Let $X = \{0_X, a, b, c\}$ and $Y = \{1, 2, 3\}$. Consider the Q-algebra $(X, *, 0_X)$ in Table 2 and $(X, \leq_X)$ in Figure 3.

$*$	0_X	a	b	c
0_X	0_X	0_X	0_X	0_X
a	a	0_X	0_X	0_X
b	b	0_X	0_X	0_X
c	c	c	c	0_X

Table 2: $(X, *, 0_X)$

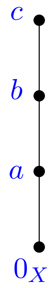

Figure 3: $(X, \leq_X)$

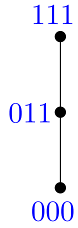

Figure 4: $(V, \leq_V)$

Define $\varphi : Y \to X$, by $\varphi = \begin{pmatrix} 1 & 2 & 3 \\ c & b & a \end{pmatrix}$, then

$$\varphi^{0_X} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \varphi^a = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}, \varphi^b = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}, \varphi^c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \varphi^{(0_X)} = \emptyset,$$
$$\varphi^{(a)} = \{2, 3\}, \varphi^{(b)} = \{2, 3\} \textit{ and } \varphi^{(c)} = \{1, 2, 3\}.$$

Then the equivalence relation R on X is as $R = \{(0, 0), (a, a), (b, b), (c, c), (a, b), (b, a)\}$. Hence, we have all distinct codewords of the binary block-code $V = \{v_{0_X} = 000, v_a = v_b = 011, v_c = 111\}$ and so have the poset $(V, \leq_V)$ in Figure 4.

Theorem 3.5. *Let $(X, *, 0_X)$ be a Q-algebra, $\varphi : Y \to X$ be a Q-valued function and $C(X) = X$. Then*

(i) if φ is surjective, then $Card(R) = Card(X)$.

(ii) if φ is surjective, then V is a linear independent subset of $\mathbb{R}^{Card(X)}$.

Theorem 3.6. *Let $(X, *, 0_X)$ be a Q-algebra, $\varphi : Y \to X$ be a surjective Q-valued function and $C(X) = X$. Then $(X, \leq_X) \cong (V, \leq_V)$.*

Proof. Define a map $f : X \to V$ by $f(w) = v_w$. Let $w, w' \in X$. Using Theorem 3.5, $R = \{(w, w) \mid w \in X\}$. It follows that for any $w \in X, R(w) = \{x \in X \mid N(w) = N(x)\} = \{w\}$ and so all codewords v_w of the binary block-code V are distinct. Then $w = w'$ if and only if $v_w = v'_w$ if and only if $f(w) = f(w')$, so f is an injective map, and so it is a bijection, because f is surjective. Since $C(X) = X$, get that $(X, *, 0_X)$ is a semigroup. Suppose $w, w' \in X$, then by Theorem 2.3, $w \leq_X w'$ implies that $\varphi^{(w)} \subseteq \varphi^{(w')}$ and using Theorem 3.3, we have $f(w) = v_w \leq_V v'_w = f(w')$. Conversely, let $f(w) \leq_V f(w')$. Applying Theorem 3.3, $\varphi^{(w)} \subseteq \varphi^{(w')}$ and so $w \leq_X w'$. Therefore, $(X, \leq_X) \cong (V, \leq_V)$. $\square$

Corollary 3.7. *Every finite Q-algebra $(X, *, 0_X)$, that $C(X) = X$ determines a binary block-code V such that $(X, \leq_X) \cong (V, \leq_V)$.*

4 Discussion of results and conclusion

Codewords in a binary block code generated by a Q-valued function are established and some interesting results are obtained. The main result is proved that every finite Q-algebra X which has the order less than or equal to the order of a finite set Y determines a binary block-code V such that $(X, \leq_X)$ is isomorphic to $(V, \leq_V)$. We consider some conditions in any Q-algebra such as the stabilizer of Q-algebra constanter of Q-algebra the set of left units of elements of Q-algebra.

Acknowledgment

The authors wish to thank the reviewers for their excellent suggestions that have been incorporated into this paper.

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