New approaches of Wolstenholme's Theorem

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Abstract

The Wolstenholme's theorem asserts that if $p \geq 3$ be a prime then the binomial coefficient $\binom{2p-1}{p-1}$ satisfies the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

In this talk, we give a family of congruences for the binomial coefficients $\binom{np-1}{p}$ modulo p^3 and modulo p^4. In fact, the first one is a conjecture that was stated by Manjil P. Saikia.

Keywords: Wolstenholme theorem; Wolstenholme-type theorem; generalized Wolstenholme's theorem, Glaisher Congruence

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1 Introduction

Wolstenholme's theorem is a result in number theory that was first conjectured by the British mathematician Joseph Wolstenholme [4] in the mid-19$^{\text{th}}$ century and later proven by Thomas Penyngton Kirkman [3] in 1862. The theorem is concerned with certain properties of binomial coefficients and their divisibility by prime numbers, which is stated as follows: For any prime number $p > 3$, the congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ holds.

It is well known that the Wolstenholm's theorem is equivalent to for any prime $p \geq 5$ the numerator of the fraction

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$$

is divisible by p^2.

In 1900, Glaisher proved generalized Wolstenholme's congruence.

Theorem 1.1 (Glaisher Congruence). *if $p \geq 5$ be a prime, we have*

$$\binom{np-1}{p-1} \equiv 1 \pmod{p^3}.$$

Glaisher and Wolstenholm's congruences, extend for higher power of prime p by some authors. The first, we need to define some notations.

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Definition 1.2. For positive integers $n \leq m$, we denote

$$R_n(m) = \sum_{k=1}^{p-1} \frac{1}{k^n} \qquad \text{and} \qquad H_n(m) = \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq m-1} \frac{1}{i_1 i_2 \cdots i_n}.$$

In 1900, *J. W. L. Glaisher* proved the following result.

Theorem 1.3 (Glaisher, [2])**.** *For any prime $p \geq 5$ we have*

$$\binom{2p-1}{p-1} \equiv 1 - 2pH_1(p-1) \equiv 1 - \frac{2}{3}p^3 B_{p-3} \pmod{p^4}.$$

Almost 5 years later, R. J. McIntosh extend Glaisher congruence to modulo p^5 and showed that for any prime $p \geq 7$, we have

$$\binom{2p-1}{p-1} \equiv 1 - p^2 R_2(p-1) \equiv 1 + 2pH_1(p-1) \pmod{p^5}.$$

Recently, Manjil P. Saikia. posed the following conjecture, which is the special case of Glaisher congruence just by replacing $p-1$ with p [1].

Conjecture 1. For a prime number $p \geq 3$ and positive integer n, the congruence

$$\binom{np-1}{p} \equiv n-1 \pmod{p^3}$$

holds.

In this note, we proved this conjecture and also generalized Saikia's conjecture for higher power of prime p and give a family of congruences for this binomial coefficient as in terms of multiple harmonic sums and generalization of the harmonic numbers.

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