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On Intuitionistic Fuzzy Operations, Modules and Homomorphisms

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Abstract

In this article, intuitionistic fuzzy binary operations on some Intuitionistic fuzzy sets are studied. We introduced Intuitionistic fuzzy groups, modules and Intuitionistic fuzzy homomorphisms under Intuitionistic fuzzy binary operation. Also we investigated some properties of Intuitionistic fuzzy group rings and modules under binary operation.

Keywords: intuitionistic fuzzy operation, intuitionistic fuzzy ring, intuitionistic fuzzy modules, intuitionistic fuzzy homomorphism

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1 Introduction

The notion of a fuzzy subset of a nonempty set S as a function from S into $[0, 1]$ was first developed by Zadeh [13] as a method of representing uncertainty in real physical world. Since then this concept has been applied to many mathematical branches. Rosenfeld [11] applied the notion of fuzzy sets to algebra and introduced the notion of fuzzy subgroups. The literature of various fuzzy algebraic concepts has been growing very rapidly. In particular, Negoita and Ralescu [10] introduced and examined the notion of fuzzy submodule of a module. Since then different types of fuzzy submodules were investigated in the last two decades. By use of Yuan and Lee's [12] definition of fuzzy group based on fuzzy binary operation, Aktas and Cagman [1] defined a new kind of fuzzy ring. One of the interesting generalizations of the theory of fuzzy sets is the theory of intuitionistic fuzzy sets introduced by Atanassov [2]. Biswas [3] was the first one to introduce the notion of intuitionistic fuzzy subgroup of a group. Using the Atanassov's idea, Davvaz et al. [4] established the intuitionistic fuzzification of the concept of submodule in a module and introduced the notion of intuitionistic fuzzy submodule of a module which was further studied by many authors.

In this study, we introduce a new kind of intuitionistic fuzzy module by using Yuan and Lee's definition of the intuitionistic fuzzy group and Aktas and Cagman's definition of fuzzy ring.

Let X be a space of points. A fuzzy set A in X is characterized by a membership function $\mu_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$ with the value of $\mu_A(x)$ at x representing the "grade of membership" of x in A.

[1] speaker

Definition 1.1. Intersection: the membership function of the intersection of two fuzzy sets A and B is defined as:

$$\mu_{A\cap B}(x) = Min(\mu_A(x), \mu_B(x)), \quad \forall x \in X$$

Definition 1.2. Union: the membership function of the union is defined as:

$$\mu_{A\cup B}(x) = Max(\mu_A(x), \mu_B(x)), \quad \forall x \in X$$

Definition 1.3. Let M be an $R-$module. The fuzzy set μ of M is called a *fuzzy submodule* (FSM) of M if

(1) $\mu(0) = 1$;

(2) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}, \forall x, y \in M$;

(3) $\mu(rx) \geq \mu(x), \forall x \in M, r \in R$.

Definition 1.4. For two fuzzy $R-$modules μ_A and μ_B; a function $f : \mu_A \longrightarrow \mu_B$ is called *fuzzy $R-$homomorphism*, if f is an $R-$homomorphism and $\mu_B(f(a)) \geq \mu_A(a)(\forall a \in A)$. For simplicity, denote by $Hom(\mu_A,\ \mu_B)$ the set of fuzzy $R-$homomorphisms from μ_A to μ_B.

Definition 1.5. Let $\theta \in [0, 1)$. Let G be a nonempty set and R be a fuzzy subset of $G \times G \times G$. R is called a *fuzzy binary operation* on G if

(1) for all $a, b \in G$, $\exists c \in G$ such that $R(a, b, c) > \theta$;

(2) for all $a, b, c_1, c_2 \in G$, $R(a, b, c_1) > \theta$ and $R(a, b, c_2) > \theta$ implies $c_1 = c_2$.

Definition 1.6. Let G be a nonempty set and R be a fuzzy binary operation on G. (G, R) is called a *fuzzy group* if the following conditions are true:

1. $\forall a, b, c, z_1, z_2 \in G,\ ((aob)oc)(z_1) > 0$ and $(ao(boc))(z_2) > 0$ implies $z_1 = z_2$;
2. $\exists e \in G$ such that $(eoa)(a) > 0$ and $(aoe)(a) > 0$ for any $a \in G$ (e is called an identity element of G);
3. $\forall a \in G, \exists b \in G$ such that $(aoB)(a) > 0$ and $(boa)(e) > 0$ (b is called an inverse dement of a and is denoted as a^{-1}).

Definition 1.7. A fuzzy set μ of a ring R is called a *fuzzy ideal*, if it satisfies the following properties:

1. $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in R$.
2. $\mu(xy) \geq \mu(x) \vee \mu(y)$, for all $x, y \in R$.

An *intuitionistic fuzzy set* (briefly an IFS) A of a non-void set X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$, where the maps $\mu_A : X \longrightarrow [0, 1]$ and $\nu_A : X \longrightarrow [0, 1]$, are fuzzy subsets of X, denote respectively the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.
For the sake of simplicity, we denote an IFS, $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$ of the set X by $A = (\mu_A, \nu_A)$ or briefly A, and the set of all IFS of X by $IFS(X)$. If X is a non-empty set and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ are two IFS of X, then
$A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$;
$A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x \in X$;

$A^c = (\nu_A, \mu_A)$;
$A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)); x \in X\}$;
$A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)); x \in X\}$.

Let $\{A_i = (\mu_{A_i}, \nu_{A_i})\}_{i \in I}$ be a family of IFS of X. Then
$\bigcap_{i \in I} A_i = (\mu_{(\cap_{i \in I} A_i)}, \nu_{(\cap_{i \in I} A_i)}) = \{(x, \bigwedge_{i \in I} \mu_{A_i}(x), \bigvee_{i \in I} \nu_{A_i}(x)); x \in X\}$ and
$\bigcup_{i \in I} A_i = (\mu_{(\cup_{i \in I} A_i)}, \nu_{(\cup_{i \in I} A_i)}) = \{(x, \bigvee_{i \in I} \mu_{A_i}(x), \bigwedge_{i \in I} \nu_{A_i}(x)); x \in X\}$

Definition 1.8. Let M be an $R-$module and $A = (\mu_A, \nu_A)$ an IFS of M. Then A is called an *intuitionistic fuzzy submodule of M* if A satisfies the following:

1. $\mu_A(0) = 1$, $\nu_A(0) = 0$
2. $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$, for all $x, y \in M$
   $\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)$, for all $x, y \in M$
3. $\mu_A(rx) \geq \mu_A(x)$, for all $x \in M$ and $r \in R$
   $\nu_A(rx) \leq \nu_A(x)$, for all $x \in M$ and $r \in R$

2 Main results

2.1 Intuitionistic Fuzzy binary operations and groups, basic properties and preliminaries

In this section we give some important definitions of intuitionistic fuzzy (IF for short) sets and operations. Then we formulate some properties and results of them.

Definition 2.1. Let $\theta \in [0, 1)$, R and S be nonempty sets and let $f = (\mu_f, \nu_f)$ be an intuitionistic fuzzy subset of $R \times S$, then A is called a (θ) intuitionistic fuzzy (IF) function from R into S if

1. $\begin{cases} \forall x \in R,\ \exists y \in s \text{ such that } \mu_f(x, y) > \theta \\ \forall x \in R,\ \exists y \in s \text{ such that } \nu_f(x, y) < 1 - \theta \end{cases}$
2. $\begin{cases} \forall x \in R \text{ for all } y_1, y_2 \in S,\ \mu_f(x, y_1) > \theta \text{ and } \mu_f(x, y_2) > \theta \text{ imply } y_1 = y_2 \\ \forall x \in R \text{ for all } y_1, y_2 \in S,\ \nu_f(x, y_1) < 1 - \theta \text{ and } \nu_f(x, y_2) < 1 - \theta \text{ imply } y_1 = y_2 \end{cases}$

Definition 2.2. Let G be a nonempty set and let $R = (\mu_R, \nu_R)$ be an IF subset of $G \times G \times G$. Then $R = (\mu_R, \nu_R)$ with $\begin{cases} \mu_R : G \times G \times G \longrightarrow [0, 1] \\ \nu_R : G \times G \times G \longrightarrow [0, 1] \end{cases}$ is called an *intuitionistic fuzzy binary operation* on G if

1. $\begin{cases} \forall a, b \in G,\ \exists c \in G \text{ such that } \mu_R(a, b, c) > \theta \\ (\forall a, b \in G,\ \exists c \in G \text{ such that } \nu_R(a, b, c) < 1 - \theta) \end{cases}$
2. $\begin{cases} \forall a, b, c_1, c_2 \in G, \mu_R(a, b, c_1) > \theta \text{ and } \mu_R(a, b, c_2) > \theta \text{ imply } c_1 = c_2 \\ \forall a, b, c_1, c_2 \in G, \nu_R(a, b, c_1) < 1 - \theta \text{ and } \nu_R(a, b, c_2) < 1 - \theta \text{ imply } c_1 = c_2 \end{cases}$

Let R be an intuitionistic fuzzy binary operation on G. Then we have a mapping

$$R : IF(G) \times IF(G) \longrightarrow IF(G),$$

$$(A, B) \longmapsto \alpha_R(A, B),$$

where $IF(G)$ is the set of all IF subsets of G, such that $\alpha_R(A,B) = (\mu_{\alpha_R}, \nu_{\alpha_R})$ where

$$\begin{cases} \mu_{\alpha_R}(A,B)(c) = \bigvee_{a,b\in G}(\mu_A(a) \wedge \mu_B(b) \wedge \mu_R(a,b,c)) \\ \nu_{\alpha_R}(A,B)(c) = \bigwedge_{a,b\in G}(\nu_A(a) \vee \nu_B(b) \vee \nu_R(a,b,c)) \end{cases}$$

Let $A = \chi^{IF}_{\{a\}} = (\chi_{\{a\}}, \chi^c_{\{a\}})$ and $B = \chi^{IF}_{\{b\}} = (\chi_{\{b\}}, \chi^c_{\{b\}})$ and let $R(A,B)$ be denoted as

$$(a \circ b)^{IF} = (\mu_{(a\circ b)}, \nu_{(a\circ b)}) \text{ and } (b \circ a)^{IF} = (\mu_{(b\circ a)}, \nu_{(b\circ a)})$$

then

$$\begin{cases} \forall c \in G, & (\mu_{(a\circ b)})(c) = \mu_R(a,b,c), \\ \forall c \in G, & (\nu_{(a\circ b)})(c) = \nu_R(a,b,c), \end{cases} \qquad \begin{cases} \forall c \in G, & (\mu_{(b\circ a)})(c) = \mu_R(b,a,c), \\ \forall c \in G, & (\nu_{(b\circ a)})(c) = \nu_R(b,a,c), \end{cases}$$

Now define

$$((a \circ b) \circ c)^{IF} = (\mu_{((a\circ b)\circ c)}, \nu_{((a\circ b)\circ c)}) \text{ and } (a \circ (b \circ c))^{IF} = (\mu_{(a\circ(b\circ c))}, \nu_{(a\circ(b\circ c))})$$

then

$$\begin{cases} \forall c \in G, & \mu_{((a\circ b)\circ c)}(z) = \bigvee_{d\in G}(\mu_R(a,b,d) \wedge \mu_R(d,c,z)) \\ \forall c \in G, & \nu_{((a\circ b)\circ c)}(z) = \bigwedge_{d\in G}(\nu_R(a,b,d) \vee \nu_R(d,c,z)) \end{cases}$$

$$\begin{cases} \forall c \in G, & \mu_{(a\circ(b\circ c))}(z) = \bigvee_{d\in G}(\mu_R(b,c,d) \wedge \mu_R(a,d,z)) \\ \forall c \in G, & \nu_{(a\circ(b\circ c))}(z) = \bigwedge_{d\in G}(\nu_R(b,c,d) \vee \nu_R(a,d,z)) \end{cases}$$

Definition 2.3. Let G be nonempty set and let $R = (\mu_R, \nu_R)$ be an IF binary operation on G. (G,R) is called an *IF group*, if the following conditions are true:

1. $\begin{cases} \forall a,b,c,z_1,z_2 \in G,\ \mu_{((a\circ b)\circ c)}(z_1) > \theta \text{ and } \mu_{(a\circ(b\circ c))}(z_2) > \theta \quad \text{imply } z_1 = z_2; \\ \forall a,b,c,z_1,z_2 \in G,\ \nu_{((a\circ b)\circ c)}(z_1) < 1-\theta \text{ and } \nu_{(a\circ(b\circ c))})(z_2) < 1-\theta \quad \text{imply } z_1 = z_2; \end{cases}$
2. there exists $e_0 \in G$, $(e_0 \circ a) = (\mu_{(e_0\circ a)},\ \nu_{(e_0\circ a)})$, $(a \circ e_0) = (\mu_{(a\circ e_0)}, \nu_{(a\circ e_0)})$ such that $\mu_{(e_0\circ a)}(a) > \theta$ and $\mu_{(a\circ e_0)}(a) > \theta$ (Consequently $\nu_{(e_0\circ a)}(a) < 1-\theta$ and $\nu_{(a\circ e_0)}(a) < 1-\theta$) for every $a \in G$ (e_0 is called an *identity element* of G).
3. For every $a \in G$, there exists $b \in G$ such that $\mu_{(a\circ b)}(e_0) > \theta$ and $\mu_{(b\circ a)}(e_0) > \theta$ (Consequently $\nu_{(a\circ b)}(e_0) < 1-\theta$ and $\nu_{(b\circ a)}(e_0) < 1-\theta$ in this case b is called an *inverse element* of a and denoted by a^{-1}.

Proposition 2.4. *The following bidirectional implications are true:*

$$\begin{cases} \mu_{((a\circ b)\circ c)}(d) > \theta \Longleftrightarrow (\mu_{(a\circ(b\circ c))})(d) > \theta; \\ \nu_{((a\circ b)\circ c)}(d) < 1-\theta \Longleftrightarrow \nu_{((a\circ b)\circ c)}(d) < 1-\theta; \end{cases}$$

Proof. Let

$$\begin{cases} \mu_{((a\circ b)\circ c)}(d) > \theta; \\ \nu_{((a\circ b)\circ c)}(d) < 1-\theta \end{cases} \text{ and let } z, w \in G \text{ such that } \begin{cases} \mu_R(b,c,z) > \theta; \\ \nu_R(b,c,z) < 1-\theta \end{cases} \text{ and } \begin{cases} \mu_R(a,z,w) > \theta; \\ \nu_R(a,z,w) < 1-\theta \end{cases}$$

Then

$$\begin{cases} \mu_{(a\circ(b\circ c))}(w) \geq \mu_R(b,c,z) \wedge \mu_R(a,z,w) > \theta; \\ \nu_{(a\circ(b\circ c))}(w) \leq \nu_R(b,c,z) \vee \nu_R(a,z,w) < 1-\theta \end{cases}$$

Thus
$d = w$ and $\begin{cases} \mu_{(a\circ(b\circ c))}(d) > \theta; \\ \nu_{(a\circ(b\circ c))}(d) < 1-\theta \end{cases}$ Similarly by $\begin{cases} \mu_{(a\circ(b\circ c))}(d) > \theta; \\ \nu_{(a\circ(b\circ c))}(d) < 1-\theta \end{cases}$ we have $\begin{cases} \mu_{((a\circ b)\circ c)}(d) > \theta; \\ \nu_{((a\circ b)\circ c)}(d) < 1-\theta \end{cases}$ ∎

Proposition 2.5. *H is an IF subgroup of G if and only if*

1. $\begin{cases} \forall a,b\in H, \forall c\in G, \mu_{(a\circ b)}(c)>\theta \text{ imlies } c\in H; \\ \forall a,b\in H, \forall c\in G, \nu_{(a\circ b)}(c)<1-\theta \text{ imlies } c\in H \end{cases}$

2. $a\in H$ implies $a^{-1}\in H$.

Definition 2.6. Let $H=(\mu_H,\nu_H)$ be an IF subgroup of G. Let

$$aH=\begin{cases} (a\mu_H)(z)=\bigvee_{x\in G}\mu_R(a,x,z); \\ (a\nu_H)(z)=\bigwedge_{x\in G}\nu_R(a,x,z). \end{cases} \qquad \text{Ha}=\begin{cases} (\mu_H a)(z)=\bigvee_{x\in G}\mu_R(x,a,z); \\ (\nu_H a)(z)=\bigwedge_{x\in G}\nu_R(x,a,z). \end{cases}$$

Then $aH(Ha)$ is called a *left(right)coset* of H.

Definition 2.7. Let $H=(\mu_H,\nu_H)$ be an IF subgroup of G. If for $(a\circ(h\circ a^{-1}))=(\mu_{(a\circ(h\circ a^{-1}))},\ \nu_{(a\circ(h\circ a^{-1}))})$

$$\begin{cases} \forall a,b\in G,\ \forall h\in H, \quad \mu_{(a\circ(h\circ a^{-1}))}(b)>\theta; \\ \forall a,b\in G,\ \forall h\in H, \quad \nu_{(a\circ(h\circ a^{-1}))}(b)<1-\theta. \end{cases}$$

then H is called a *normal IF subgroup* of G.

Definition 2.8. Let (G,R) be an IF subgroup. If

$$\mu_{(a\circ b)}(c)>\theta \Longleftrightarrow \mu_{(b\circ a)}(c)>\theta, \quad \forall a,b,c\in G$$

$$\nu_{(a\circ b)}(c)<1-\theta \Longleftrightarrow \nu_{(b\circ a)}(c)<1-\theta, \quad \forall a,b,c\in G$$

then $(G,R)^{IF}$ is called an *abelian IF group.*

Theorem 2.9. *Let* $[aH]=\{a'H \mid a'H\sim aH\}$,
$\bar{a}=\{a' |\ a'\in G \text{ and } a'H\sim aH\},\ G/H=\{[aH]|\ a\in G\}$, *and*

$$\bar{R}=(\mu_{\bar{R}},\nu_{\bar{R}})=\begin{cases} \mu_{\bar{R}}:\frac{G}{H}\times\frac{G}{H}\times\frac{G}{H}\longrightarrow[0,1], \\ \nu_{\bar{R}}:\frac{G}{H}\times\frac{G}{H}\times\frac{G}{H}\longrightarrow[0,1]. \end{cases}$$

$$([aH],[bH],[cH])\longmapsto\bar{R}([aH],[bH],[cH])=\begin{cases} \bigvee\limits_{(a',b',c')\in\bar{a}\times\bar{b}\times\bar{c}}\mu_R(a',b',c'), \\ \bigwedge\limits_{(a',b',c')\in\bar{a}\times\bar{b}\times\bar{c}}\nu_R(a',b',c'). \end{cases}$$

Then $\bar{R}$ is an IF binary relation on $\frac{G}{H}$.

Proof.

(1) $\forall a,b\in G,\ \exists c\in G$ such that $\mu_R(a,b,c)>\theta$, then

$$\begin{cases} \mu_{\bar{R}}([aH],[bH],[cH])\geq\mu_R(a,b,c)>\theta, \\ \nu_{\bar{R}}([aH],[bH],[cH])\leq\mu_R(a,b,c)<1-\theta. \end{cases}$$

(2) Let

$$M=\begin{cases} \mu_{\bar{R}}([aH],[bH],[cH])>\theta, \\ \nu_{\bar{R}}([aH],[bH],[cH])<1-\theta. \end{cases}$$

and

$$N=\begin{cases} \mu_{\bar{R}}([aH],[bH],[dH])>\theta, \\ \nu_{\bar{R}}([aH],[bH],[dH])<1-\theta. \end{cases}$$

We need to prove $[cH] = [dH]$.
There exist $a_1 \in \bar{a},\ b_1 \in \bar{b},\ c_1 \in \bar{c},\ a'_1 \in \bar{a},\ b'_1 \in \bar{b},\ d_1 \in d$ such that

$$\begin{cases} \mu_R(a_1, b_1, c_1) > \theta & \mu_R(a'_1, b'_1, c'_1) > \theta, \\ \nu_R(a_1, b_1, c_1) < 1-\theta & \nu_R(a'_1, b'_1, c'_1) < 1-\theta. \end{cases}$$

Since $a'_1H \sim a_1H,\ b'_1H \sim b_1H$, so there exist $h_1 \in H, h_2 \in H$ such that

$$\begin{cases} \mu_R(a'_1, h_1, a_1) > \theta & \mu_R(b'_1, h_2, b_1) > \theta, \\ \nu_R(a'_1, h_1, a_1) < 1-\theta & \nu_R(b'_1, h_2, b_1) < 1-\theta. \end{cases}$$

Let $z \in G$ such that $\begin{cases} \mu_R(h_1, b'_1, z) > \theta, \\ \nu_R(h_1, b'_1, z) < 1-\theta. \end{cases}$, then $\begin{cases} \mu_R(z, {b'_1}^{-1}, h_1) > \theta, \\ \nu_R(z, {b'_1}^{-1}, h_1) < 1-\theta. \end{cases}$,

So ${b'_1}^{-1}H \sim zH$ and there exists $h'_1 \in H$ such that $\begin{cases} \mu_R({b'_1}^{-1}, z, h') > \theta, \\ \nu_R({b'_1}^{-1}, z, h') < 1-\theta. \end{cases}$.

Let $y \in G$ such that $\begin{cases} \mu_R(b'_1, h'_1, y) > \theta, \\ \nu_R(b'_1, h'_1, y) < 1-\theta. \end{cases}$, then for
$(b'_1 \circ ({b'_1}^{-1} \circ z))^{IF} = (\mu_{(b'_1 \circ ({b'_1}^{-1} \circ z))}, \nu_{(b'_1 \circ ({b'_1}^{-1} \circ z))})$,

$$\begin{cases} \mu_{(b'_1 \circ ({b'_1}^{-1} \circ z))}(y) \geq \mu_R({b'_1}^{-1}, z, h'_1) \wedge \mu_R(b'_1, h'_1, y) > \theta, \\ \nu_{(b'_1 \circ ({b'_1}^{-1} \circ z))}(y) \leq \nu_R({b'_1}^{-1}, z, h'_1) \vee \nu_R(b'_1, h'_1, y) < 1-\theta \end{cases}$$

and $((b'_1 \circ {b'_1}^{-1}) \circ z)^{IF} = (\mu_{((b'_1 \circ {b'_1}^{-1}) \circ z)}, \nu_{((b'_1 \circ {b'_1}^{-1}) \circ z)})$,

$$\begin{cases} \mu_{((b'_1 \circ {b'_1}^{-1}) \circ z)}(z) \geq \mu_R(b'_1, {b'_1}^{-1}, e) \wedge \mu_R(e, z, z) > \theta, \\ \nu_{((b'_1 \circ {b'_1}^{-1}) \circ z)}(z) \leq \nu_R(b'_1, {b'_1}^{-1}, e) \vee \nu_R(e, z, z) < 1-\theta \end{cases}$$

Thus, $y = z$. Let $z_1, y_1 \in G$ such that
$\mu_R(h_1, b_1, z_1) > \theta,$
$\nu_R(h_1, b_1, z_1) < 1-\theta.$,
$\mu_R(a'_1, z_1, y_1) > \theta,$
$\nu_R(a'_1, z_1, y_1) < 1-\theta.$,
then $(a'_1 \circ (h_1 \circ b_1))^{IF} = (\mu_{(a'_1 \circ (h_1 \circ b_1))}, \nu_{(a'_1 o (h_1 \circ b_1))})$

$$\begin{cases} \mu_{(a'_1 \circ (h_1 \circ b_1))}(y_1) \geq \mu_R(h_1, b_1, z_1) \wedge \mu_R(a'_1, z_1, y_1) > \theta, \\ \nu_{(a'_1 \circ (h_1 \circ b_1))}(y_1) \leq \nu_R(h_1, b_1, z_1) \vee \nu_R(a'_1, z_1, y_1) < 1-\theta \end{cases}$$

$((a'_1 \circ h_1) \circ b_1)^{IF} = (\mu_{((a'_1 o h_1) \circ b_1)}, \nu_{((a'_1 o h_1) \circ b_1)})$

$$\begin{cases} \mu_{((a'_1 \circ h_1) \circ b_1)} \geq \mu_R(h_1, b_1, z_1) \wedge \mu_R(a'_1, z_1, y_1) > \theta, \\ \nu_{((a'_1 \circ h_1) \circ b_1)} \leq \nu_R(h_1, b_1, z_1) \vee \nu_R(a'_1, z_1, y_1) < 1-\theta \end{cases}$$

Thus, $y_1 = c_1$ and $\begin{cases} \mu_R(a'_1, z_1, c_1) > \theta, \\ \nu_R(a'_1, z_1, c_1) < 1-\theta \end{cases}$

Let $p_1 \in G$ such that $\begin{cases} \mu_R(z, h_2, p_1) > \theta, \\ \nu_R(z, h_2, p_1) < 1-\theta \end{cases}$ then for
$((h_1 \circ b'_1) \circ h_2)^{IF} = (\mu_{((h_1 \circ b'_1) o h_2)}, \nu_{((h_1 \circ b'_1) \circ h_2)})$

$$\begin{cases} \mu((h_1 \circ b'_1) \circ h_2)(p_1) \geq \mu_R(h_1, b'_1, z) \wedge \mu_R(z, h_2, p_1) > \theta, \\ \nu((h_1 \circ b'_1) \circ h_2)(p_1) \leq \nu_R(h_1, b'_1, z) \vee \nu_R(z, h_2, p_1) < 1-\theta \end{cases}$$

and for $(h_1 \circ (b'_1 \circ h_2))^{IF} = (\mu_{(h_1 \circ (b'_1 \circ h_2))}, \nu_{(h_1 \circ (b'_1 \circ h_2))})$ so

$$\begin{cases} \mu_{(h_1 \circ (b'_1 \circ h_2))}(z_1) \geq \mu_R(b'_1, h_2, b_1) \wedge \mu_R(h_1, b_1, z) > \theta, \\ \nu_{(h_1 \circ (b'_1 o h_2)}(z_1) \leq \nu_R(b'_1, h_2, b_1) \vee \nu_R(h_1, b_1, z) < 1-\theta \end{cases}$$

Thus, $p_1 = z_1$ and $\left\{\begin{array}{c}\mu_R(z, h_2, z_1) > \theta, \\ \nu_R(z, h_2, z_1) < 1-\theta\end{array}\right.$, $\left\{\begin{array}{c}\mu_R(y, h_2, z_1) > \theta, \\ \nu_R(y, h_2, z_1) < 1-\theta\end{array}\right.$.

Let $h \in G$, $w_1 \in G$ such that $\left\{\begin{array}{c}\mu_R(h'_1, h_2, h) > \theta, \\ \nu_R(h'_1, h_2, h) < 1-\theta\end{array}\right.$, $\left\{\begin{array}{c}\mu_R(b'_1, h, w_1) > \theta, \\ \nu_R(b'_1, h, w_1) < 1-\theta\end{array}\right.$,

then $h \in H$ and $(b'_1 \circ (h'_1 \circ h_2)^{IF} = (\mu_{(b'_1 \circ (h'_1 \circ h_2)}, \nu_{(b'_1 \circ (h'_1 \circ h_2)})$ so

$$\left\{\begin{array}{c}\mu_{(b'_1 \circ (h'_1 \circ h_2)}(w_1) \geq \mu_R(h'_1, h_2, h) \wedge \mu_R(b'_1, h, w_1) > \theta, \\ \nu_{(b'_1 \circ (h'_1 \circ h_2)}(w_1) \leq \nu_R(h'_1, h_2, h) \vee \nu_R(b'_1, h, w_1) < 1-\theta\end{array}\right.$$

$((b'_1 \circ h'_1) \circ h_2)^{IF} = (\mu_{((b'_1 \circ h'_1) \circ h_2)}, \nu_{((b'_1 \circ h'_1) \circ h_2)})$, hence

$$\left\{\begin{array}{c}\mu_{((b'_1 \circ h'_1) \circ h_2)}(z_1) \geq \mu_R(b'_1, h'_1, y) \wedge \mu_R(y, h_2, z_1) > \theta, \\ \nu_{((b'_1 \circ h'_1) \circ h_2)}(z_1) \leq \nu_R(b'_1, h'_1, y) \vee \nu_R(y, h_2, z_1) < 1-\theta\end{array}\right.$$

Thus, $w_1 = z_1$ and $\left\{\begin{array}{c}\mu_R(b'_1, h, z_1) > \theta, \\ \nu_R(b'_1, h, z_1) < 1-\theta\end{array}\right.$

Let $w \in G$ such that $\left\{\begin{array}{c}\mu_R(d_1, h, w) > \theta, \\ \nu_R(d_1, h, w) < 1-\theta\end{array}\right.$, then $((a'_1 \circ b'_1) \circ h)^{IF} = (\mu_{((a'_1 \circ b'_1) \circ h)}, \nu_{((a'_1 \circ b'_1) \circ h)})$ so

$$\left\{\begin{array}{c}\mu_{((a'_1 \circ b'_1) \circ h)}(w) \geq \mu_R(a'_1, b'_1, d_1) \wedge \mu_R(d_1, h, w) > \theta, \\ \nu_{((a'_1 \circ b'_1) \circ h)}(w) \leq \nu_R(a'_1, b'_1, d_1) \vee \nu_R(d_1, h, w) < 1-\theta\end{array}\right.$$

$(a'_1 \circ (b'_1 \circ h))^{IF} = (\mu_{(a'_1 \circ (b'_1 \circ h))}, \nu_{(a'_1 \circ (b'_1 \circ h))})$ implies

$$\left\{\begin{array}{c}\mu_{(a'_1 \circ (b'_1 \circ h))}(c_1) \geq \mu_R(b'_1, h, z_1) \wedge \mu_R(a'_1, z_1, c_1) > \theta, \\ \nu_{(a'_1 \circ (b'_1 \circ h))}(c_1) \leq \nu_R(b'_1, h, z_1) \vee \nu_R(a'_1, z_1, c_1) < 1-\theta\end{array}\right.$$

Thus, $w = c_1$ and $\left\{\begin{array}{c}\mu_R(d_1, h, c_1) > \theta, \\ \nu_R(d_1, h, c_1) < 1-\theta\end{array}\right.$.

It follows that $cH \sim dH$ and consequently $[cH] = [dH]$.
Hence, $\bar{R}$ is an IF binary operation on $\frac{G}{H}$. Since $\bar{R}$ is an IF binary operation on $\frac{G}{H}$, so we have $([aH] \circ [bH])^{IF} = (\mu_{([aH] \circ [bH])}, \nu_{([aH] \circ [bH])})$ so,

$$\left\{\begin{array}{c}\mu_{([aH] \circ [bH])}([cH]) = \mu_{\bar{R}}([aH], [bH], [cH]), \\ \nu_{([aH] \circ [bH])}([cH]) = \nu_{\bar{R}}([aH], [bH], [cH])\end{array}\right.$$

and $(([aH] \circ [bH]) \circ [cH])^{IF} = (\mu_{(([aH] \circ [bH]) \circ [cH])}, \nu_{(([aH] \circ [bH]) \circ [cH])})$ implies

$$\left\{\begin{array}{c}\mu_{(([aH] \circ [bH]) \circ [cH])}([dh]) = \bigvee \mu_{\bar{R}}([aH], [bH], [xH]) \wedge \mu_{\bar{R}}([xH], [cH], [dH]), \\ \nu_{(([aH] \circ [bH]) \circ [cH])}([dh]) = \bigwedge \nu_{\bar{R}}([aH], [bH], [xH]) \vee \nu_{\bar{R}}([xH], [cH], [dH])\end{array}\right.$$

$([aH] \circ ([bH] \circ [cH])^{IF} = (\mu_{([aH] \circ ([bH] \circ [cH])}, \nu_{([aH] \circ ([bH] \circ [cH])})$ implies

$$\left\{\begin{array}{c}\mu_{([aH] \circ ([bH] \circ [cH])}([wh]) = \bigvee \mu_{\bar{R}}([aH], [cH], [xH]) \wedge \mu_{\bar{R}}([aH], [xH], [wH]), \\ \nu_{([aH] \circ ([bH] \circ [cH])}([wh]) = \bigwedge \nu_{\bar{R}}([aH], [cH], [xH]) \vee \nu_{\bar{R}}([aH], [xH], [wH]).\end{array}\right.$$

□

Theorem 2.10. *$(\frac{G}{H}, \alpha_{\bar{R}})$ is an IF group.*

Proof. Let

$$(([aH]\circ[bH])\circ[cH])^{IF}=(\mu_{(([aH]\circ[bH])\circ[cH])},\nu_{(([aH]\circ[bH])\circ[cH])})$$

and

$$([aH]\circ([bH]\circ[cH]))^{IF}=(\mu_{([aH]\circ([bH]\circ[cH]))},\nu_{([aH]\circ([bH]\circ[cH]))}).$$

It implies

$$\begin{cases}\mu_{(([aH]\circ[bH])\circ[cH])}([dh])>\theta, & \mu_{([aH]\circ([bH]\circ[cH]))}([wH])>\theta,\\ \nu_{(([aH]\circ[bH])\circ[cH])}([dh])<1-\theta, & \nu_{([aH]\circ([bH]\circ[cH]))}([wH])<1-\theta\end{cases}$$

Then, we have $a_1,a_1',b_1,b_1',c_1,c_1',w_1\in G$ such that
$c_1H\sim c_1'H\sim cH,\ a_1'H\sim a_1H\sim aH,\ b_1'H\sim b_1H\sim bH,\ d_1H\sim dH,\ w_1H\sim wH$
and there exist elements $h_1,h_2,h_3\in H,\ x_1',x_2'\in G$ such that

$$\begin{cases}\mu_R(a_1,b_1,x_1')\wedge\mu_R(x_1',c_1,d_1)>\theta,\\ \nu_R(a_1,b_1,x_1')\vee\nu_R(x_1',c_1,d_1)<1-\theta\end{cases}$$

$$\begin{cases}\mu_R(b_1',c_1',x_2')\wedge\mu_R(a_1',x_2',w_1)>\theta,\\ \nu_R(b_1',c_1',x_2')\vee\nu_R(a_1',x_2',w_1)<1-\theta\end{cases}$$

$$\begin{cases}\mu_R(a_1',h_1,a_1)>\theta & \mu_R(b_1',h_2,b_1)>\theta & \mu_R(c_1',h_3,c_1)>\theta,\\ \nu_R(a_1',h_1,a_1)<1-\theta & \nu_R(b_1',h_2,b_1)<1-\theta & \nu_R(c_1',h_3,c_1)<1-\theta\end{cases}$$

Let $z_1\in G$ such that $\begin{cases}\mu_R(a_1',b_1',z_1)>\theta,\\ \nu_R(a_1',b_1',z_1)<1-\theta\end{cases}$, then by

$$\begin{cases}\mu_R(a_1,b_1,x_1')>\theta,\\ \nu_R(a_1,b_1,x_1')<1-\theta\end{cases},\begin{cases}\mu_R(a_1',h_1,a_1)>\theta,\\ \nu_R(a_1',h_1,a_1)<1-\theta\end{cases},\begin{cases}\mu_R(a_1',b_1',z_1)>\theta,\\ \nu_R(a_1',b_1',z_1)<1-\theta\end{cases},\begin{cases}\mu_R(b_1',h_2,b_1)>\theta,\\ \nu_R((b_1',h_2,b_1)<1-\theta\end{cases}$$

and the proof of theorem 2.9, there exists $h_4\in H$ such that $\begin{cases}\mu_R(z_2,h_4,d_1)>\theta,\\ \nu_R(z_2,h_4,d_1)<1-\theta\end{cases}$.

$$(a_1'\circ(b_1'\circ c_1'))^{IF}=(\mu_{(a_1'\circ(b_1'\circ c_1'))},\ \nu_{(a_1'\circ(b_1'\circ c_1'))})$$

then

$$\begin{cases}\mu_{(a_1'\circ(b_1'\circ c_1'))}(w_1)>\mu_R(b_1',c_1',x_2')\wedge\mu_R(a_1',x_2',w_1)>\theta,\\ \nu_{(a_1'\circ(b_1'\circ c_1'))}(w_1)<\nu_R(b_1',c_1',x_2')\vee\nu_R(a_1',x_2',w_1)<1-\theta\end{cases}$$

$$((a_1'\circ b_1')\circ c_1')^{IF}=(\mu_{((a_1'\circ b_1')\circ c_1')},\nu_{((a_1'\circ b_1')\circ c_1')})$$

then

$$\begin{cases}\mu_{((a_1'\circ b_1')\circ c_1')}(w_1)>\mu_R(a_1,b_1',z_1)\wedge\mu_R(z_1,c_1',z_2)>\theta,\\ \nu_{((a_1'\circ b_1')\circ c_1')}(w_1)<\nu_R(a_1,b_1',z_1)\vee\nu_R(z_1,c_1',z_2)<1-\theta\end{cases}$$

so, $z_2=w_1$ and $\begin{cases}\mu_R(w_1,h_4,d_1)>\theta,\\ \nu_R(w_1,h_4,d_1)<1-\theta\end{cases}$. Then, $w_1H\sim dH$ and consequently $[wH]=[dH]$.

$([aH]\circ[eH])^{IF}=(\mu_{([aH]\circ[eH])},\nu_{([aH]\circ[eH])})$ and $([eH]\circ[aH])^{IF}=(\mu_{([eH]\circ[aH])},\nu_{([eH]\circ[aH])})$ implies

$$\begin{cases}\forall a\in G,\mu_{([aH]\circ[eH])}([aH])\geq\mu_R(a,e,a)>\theta,\ \mu_{([eH]\circ[aH])}([aH])\geq\mu_R(e,a,a)>\theta\\ \forall a\in G,\nu_{([aH]\circ[eH])}([aH])\leq\nu_R(a,e,a)<1-\theta,\ \nu_{([eH]\circ[aH])}([aH])\leq\nu_R(e,a,a)<1-\theta\end{cases}$$

$$([aH] \circ [a^{-1}H])^{IF} = (\mu_{([aH]\circ[a^{-1}H])}, \nu_{([aH]\circ[a^{-1}H])})$$

and

$$([a^{-1}H] \circ [aH])^{IF} = (\mu_{([a^{-1}H]\circ[aH])}, nu_{([a^{-1}H]\circ[aH])})$$

imply

$$\begin{cases} \mu_{([aH]\circ[a^{-1}H])}([eH]) \geq \mu_R(a, a^{-1}, e) > \theta, \ \mu_{([a^{-1}H]\circ[aH])}([eH]) \geq \mu_R(a^{-1}, a, e) > \theta \\ \nu_{([aH]\circ[a^{-1}H])}([eH]) \leq \nu_R(a, a^{-1}, e) < 1-\theta, \ \nu_{([a^{-1}H]\circ[aH])}([eH]) \leq \nu_R(a^{-1}, a, e) < 1-\theta \end{cases}$$

Hence, $(\frac{G}{H}, \bar{R})$ is IF group. □

Definition 2.11. Let (G_1, R_1) and (G_2, R_2) be two IF group and let $f : G_1 \longrightarrow G_2$ be a mapping. If

$$\mu_{R_1}(a, b, c) > \theta \Longrightarrow \mu_{R_2}(f(a), f(b), f(c)) > \theta$$

$$\nu_{R_1}(a, b, c) < 1-\theta \Longrightarrow \nu_{R_2}(f(a), f(b), f(c)) < 1-\theta$$

then f is called an *IF (group) homomorphism.* IF f is 1-1, it is called an IF epimorphism. If f is both 1-1 and onto, it is called an IF isomorphism.

Let $G = (\mu_G, \nu_G)$ be an IF binary operation on R. Then we have a mapping

$$\alpha_G : IF(R) \times IF(R) \longrightarrow IF(R)$$

$$(A, B) \longmapsto \alpha_G(A, B)$$

where $IF(R) = \left\{ A = (\mu_A, \nu_A) | \begin{array}{l} \mu_A : R \to [0,1] \\ \nu_A : R \to [0,1] \end{array} \right.$ is a mapping and

$$\mu_G(A, B)(c) = \bigvee_{a,b\in R} (\mu_A(a) \wedge \mu_B(b) \wedge \mu_G(a, b, c))$$

$$\nu_G(A, B)(c) = \bigwedge_{a,b\in R} (\nu_A(a) \vee \nu_B(b) \vee \nu_G(a, b, c))$$

Let $A = \chi^{IF}_{\{A\}} = (\chi_{\{A\}}, \chi^C_{\{A\}})$ and $B = \chi^{IF}_{\{B\}} = (\chi_{\{B\}}, \chi^c_{\{B\}})$ and Let $G(A, B)$ and $H(A, B)$ be denoted as $a \circ b$ and $a * b$, respectively. Then for $a \circ b$ and $a * b = (\mu_{a*b}, \nu_{a*b})$

$$\mu_{(a\circ b)}(c) = \mu_G(a, b, c) \ , \quad \forall c \in R,$$

$$\nu_{(a\circ b)}(c) = \nu_G(a, b, c) \ , \quad \forall c \in R.$$

$$\mu_{(a*b)}(c) = \mu_H(a, b, c) \ , \quad \forall c \in R,$$

$$nu_{(a*b)}(c) = \nu_H(a, b, c) \ , \quad \forall c \in R.$$

$$\begin{cases} \mu_{((a\circ b)\circ c)}(z) = \bigvee_{d\in G}(\mu_G(a, b, d) \wedge \mu_G(d, c, z)) \\ \nu_{((a\circ b)\circ c)}(z) = \bigwedge_{d\in G}(\nu_G(a, b, d) \vee \nu_G(d, c, z)) \end{cases}$$

$$\begin{cases} \mu_{(a\circ(b\circ c))}(z) = \bigvee_{d\in G}(\mu_G(b, c, d) \wedge \mu_G(a, d, z)) \\ \nu_{(a\circ(b\circ c))}(z) = \bigwedge_{d\in G}(\nu_G(b, c, d) \vee \nu_G(a, d, z)) \end{cases}$$

$$a * (b \circ a) = (\mu_{a*(b\circ a)}, \nu_{a*(b\circ a)}) \Rightarrow \begin{cases} \mu_{(a*(b\circ c))}(z) = \bigvee_{d\in G}(\mu_G(b, c, d) \wedge \mu_H(a, d, z)) \\ \nu_{(a*(b\circ c))}(z) = \bigwedge_{d\in G}(\nu_G(b, c, d) \vee \nu_H(a, d, z)) \end{cases}$$

$$\text{for } ((a*b)\circ(a*c)) = (\mu_{((a*b)\circ(a*c))}, \nu_{((a*b)\circ(a*c))})$$

implies

$$\begin{cases} \mu_{((a*b)\circ(a*c))}(z) = \bigvee_{d\in G}(\mu_H(a,b,d)\wedge\mu_H(a,c,e)\wedge\mu_G(d,e,z)) \\ \nu_{((a*b)\circ(a*c))}(z) = \bigwedge_{d\in G}(\nu_H(a,b,d)\vee\nu_H(a,c,e)\vee\nu_G(d,e,z)) \end{cases}$$

Definition 2.12. Let R be a nonempty set and let G and H be two IF binary operations on R. Then (R,G,H) is called *IF ring* if the following conditions hold for $((a*b)*c)^{IF} = (\mu_{((a*b)*c)},\ \nu_{((a*b)*c)})$, $(a*(b*c))^{IF} = (\mu_{(a*(b*c))}, \nu_{(a*(b*c))})$, $((aob)*c)^{IF} = (\mu_{((a\circ b)*c)}, \nu_{((a\circ b)*c)})$, $((a*c)\circ(b*c))^{IF} = (\mu_{((a*c)\circ(b*c))}, \nu_{((a*c)\circ(b*c))})$ and $((a*b)\circ c)^{IF} = (\mu_{((a*b)\circ c)}, \nu_{((a*b)\circ c)})$

1. (R,G) is an abelian IF group;
2. $\begin{cases} \forall a,b,c,z_1,z_2\in R,\ \mu_{((a*b)*c)}(z_1) > \theta \text{ and } \mu_{(a*(b*c))}(z_2) > \theta \\ \forall a,b,c,z_1,z_2\in R,\ \nu_{((a*b)*c)}(z_1) < 1-\theta \text{ and } \nu_{(a*(b*c))}(z_2) < 1-\theta \end{cases}$ imply $z_1 = z_2$
3. $\begin{cases} \forall a,b,c,z_1,z_2\in R,\ \mu_{((a\circ b)*c)}(z_1) > \theta \text{ and } \mu_{((a*c)\circ(b*c))}(z_2) > \theta \\ \forall a,b,c,z_1,z_2\in R,\ \nu_{((a\circ b)*c)}(z_1) < 1-\theta \text{ and } \nu_{((a*c)\circ(b*c))}(z_2) < 1-\theta \end{cases}$ imply $z_1 = z_2$
4. $\begin{cases} \forall a,b,c,z_1,z_2\in R,\ \mu_{((a*b)\circ c)}(z_1) > \theta \text{ and } \mu_{((a*b)\circ(a*c))}(z_2) > \theta \\ \forall a,b,c,z_1,z_2\in R,\ \nu_{((a*b)\circ c)}(z_1) < 1-\theta \text{ and } \nu_{((a*b)\circ(a*c))}(z_2) < 1-\theta \end{cases}$ imply $z_1 = z_2$

Definition 2.13. Let (R,G,H) be a IF ring.

1. If $\begin{cases} \mu_{(a*b)}(u) > \theta \Longleftrightarrow \mu_{(b*a)}(u) > \theta \\ \nu_{(a*b)}(u) < 1-\theta \Longleftrightarrow \nu_{(b*a)}(u) < 1-\theta \end{cases}$ then (R,G,H) is said to be a commutative IF ring.
2. If $\exists e_* \in R$ such that for $(a*e_*)^{IF} = (\mu_{(a*e_*)}, \nu_{(a*e_*)})$, $(e_* * a)^{IF} = (\mu_{(e_* * a)},\ \nu_{(e_* * a)})$ $\begin{cases} \mu_{(a*e_*)}(a) > \theta \\ \nu_{(a*e_*)}(a) < 1-\theta \end{cases}$ and $\begin{cases} \mu_{(e_* * a)}(a) > \theta \\ \nu_{(e_* * a)}(a) < 1-\theta \end{cases}$ for every $a \in R$, then (R,G,H) is said to be IF ring with identity.
3. Let (R,G,H) be an IF ring with identity. If for a member $a \in R$ there exists $b \in R$ such that $\begin{cases} \mu_{(a*b)}(e_*) > \theta \\ \nu_{(a*b)}(e_*) < 1-\theta \end{cases}$ and $\begin{cases} \mu_{(b*a)}(e_*) > \theta \\ \nu_{(b*a)}(e_*) < 1-\theta \end{cases}$, then b is said to be an inverse element of a and is denoted by a^{-1}.

Proposition 2.14. *Let (R,G,H) be an IF ring and let S be an nonempty subset of R. Then (S,G,H) is an IF subring of R if and only if*

1. *for all $a,b\in S,\ c\in R$* $\begin{cases} \mu_{(a\circ b)}(c) > \theta \\ \nu_{(a\circ b)}(c) < 1-\theta \end{cases}$ implies $c \in S$ and $\begin{cases} \mu_{(a*b)}(c) > \theta \\ \nu_{(a*b)}(c) < 1-\theta \end{cases}$ implies $c \in S$;
2. *$a \in S$ implies $a^{-1} \in S$*

Definition 2.15. A nonempty subset $I = (\mu_I, \nu_I)$ of a IF ring (R,G,H) is called a *IF ideal* of R if the following conditions are satisfied:

1. $\begin{cases} \forall x,y\in I,\ \mu_{(xoy)}(z) > \theta \\ \forall x,y\in I,\ \nu_{(xoy)}(z) < 1-\theta \end{cases} \Longrightarrow z \in I$ for all $z \in R$
2. $\{\ \forall x \in I,\ x^{-1} \in I;$
3. $\begin{cases} \forall s\in I,\ \text{for all } r\in R,\ \mu_{(r*s)}(x) > \theta \Longrightarrow x\in I \text{ and } \mu_{(s*r)}(y) > \theta \\ \forall s\in I,\ \text{for all } r\in R,\ \nu_{(r*s)}(x) < 1-\theta \Longrightarrow x\in I \text{ and } \nu_{(s*r)}(y) < 1-\theta \end{cases} \Longrightarrow y \in I,\ x,\ y \in R$

2.2 IF Modules over IF Rings

Let $(R,\ G,\ H)$ be a IF ring and $(M,\ J)$ be an abelian IF group and let α_P be IF function $R\times M$ into M. Then we have a mapping

$$P:\ IF(R)\times IF(M)\longrightarrow IF(M)$$

$$(A,N)\longmapsto P(A,N)$$

$$P=(\mu_P,\ \nu_P)\Rightarrow\begin{cases}\mu_P(A,N)(x)=\bigvee\limits_{(r,n)\in A\times N}(A(r)\wedge N(n)\wedge p(r,n,x)),\\ \nu_P(A,N)(x)=\bigwedge\limits_{(r,n)\in A\times N}(A(r)\vee N(n)\vee p(r,n,x)),\end{cases}$$

where $IF(R)=\left\{A=(\mu_A,\nu_A)\mid \begin{array}{l}\mu_A:R\longrightarrow[0,1]\\ \nu_A:R\longrightarrow[0,1]\end{array}\right\}$ and $IF(M)=\left\{N=(\mu_N,\nu_N)\mid \begin{array}{l}\mu_N:M\longrightarrow[0,1]\\ \nu_N:M\longrightarrow[0,1]\end{array}\right\}$.

Let $A=\{r\}$ and $N=\{M\}$, and let $P(A,N)$ and $J(a,b)$ be denoted as $r\odot m$ and $a\oplus b$, respectively. Then

$$(r\odot m)(x)=P(r,m,x),\quad\forall x\in M,$$

$$(r\odot(m_1\oplus m_2))^{IF}=(\mu_{(r\odot(m_1\oplus m_2))},\nu_{(r\odot(m_1\oplus m_2))})\Rightarrow\begin{cases}\mu_{(r\odot(m_1\oplus m_2))}(x)=\bigvee\limits_{m\in M}(\mu_J(m_1,m_2,m)\wedge\mu_p(r,m,x))\\ \nu_{(r\odot(m_1\oplus m_2))}(x)=\bigwedge\limits_{m\in M}(\nu_J(m_1,m_2,m)\vee\nu_p(r,m,x))\end{cases}$$

$$\begin{cases}((r_1\circ r_2)\odot m)^{IF}=(\mu_{((r_1\circ r_2)\odot m)},\nu_{((r_1\circ r_2)\odot m)}),\\ ((r_1*r_2)\odot m)^{IF}=(\mu_{((r_1*r_2)\odot m)},\ \nu_{((r_1*r_2)\odot m)})\\ ((r_1\odot(r_2\odot m))^{IF}=(\mu_{((r_1\odot(r_2\odot m))},\ \nu_{((r_1\odot(r_2\odot m))})\end{cases}$$

imply

$$\begin{cases}\mu_{((r_1\circ r_2)\odot m)}(x)=\bigvee\limits_{r\in R}(\mu_G(r_1,r_2,r)\wedge\mu_P(r,m,x))\\ \nu_{((r_1\circ r_2)\odot m)}(x)=\bigwedge\limits_{r\in R}(\nu_G(r_1,r_2,r)\vee\nu_P(r,m,x))\end{cases}$$

$$\begin{cases}\mu_{((r_1*r_2)\odot m)}(x)=\bigvee\limits_{r\in R}(\mu_H(r_1,r_2,r)\wedge\mu_P(r,m,x))\\ \nu_{((r_1*r_2)\odot m)}(x)=\bigwedge\limits_{r\in R}(\nu_H(r_1,r_2,r)\vee\nu_P(r,m,x))\end{cases}$$

$$\begin{cases}\mu_{((r_1\odot(r_2\odot m))}(x)=\bigvee\limits_{m_1\in M}(\mu_P(r_2,m,m_1)\wedge\mu_P(r_1,m_1,x))\\ \nu_{((r_1\odot(r_2\odot m))}(x)=\bigwedge\limits_{m_1\in M}(\nu_P(r_2,m,m_1)\vee\nu_P(r_1,m_1,x))\end{cases}$$

$(r\odot(m_1\oplus m_2))^{IF}=(\mu_{(r\odot(m_1\oplus m_2))},\nu_{(r\odot(m_1\oplus m_2))})$ implies

$$\begin{cases}\mu_{((r\odot m_1)\oplus(r\odot m_2))}(x)=\bigvee\limits_{x_1,x_2\in M}(\mu_P(r,m_1,x_1)\wedge\mu_P(r,m_2,x_2)\wedge\mu_J(x_1,x_2,x))\\ \nu_{((r\odot m_1)\oplus(r\odot m_2))}(x)=\bigwedge\limits_{x_1,x_2\in M}(\nu_P(r,m_1,x_1)\vee\nu_p(r,m_2,x_2)\vee\nu_J(x_1,x_2,x))\end{cases}$$

Definition 2.16. Let (R,G,H) be an IF ring and Let (M,J) be an abelian IF group. M is called an (left) IF module over R or (left) $R-$IFmodule together with an IF function $P:R\times M\longrightarrow M$, if the following conditions hold, for all $r,r_1,r_2\in R$ and for all $m,m_1,m_2\in M$,
denote $(r\odot(m_1\oplus m_2))^{IF}=(\mu_{(r\odot(m_1\oplus m_2))},\nu_{(r\odot(m_1\oplus m_2))})$, it implies

- $\begin{cases}\mu_{(r\odot(m_1\oplus m_2))}(x)>\theta \text{ and } \mu_{((r\odot m_1)\oplus(r\odot m_2))}(y)>\theta \text{ imply } x=y\\ \nu_{(r\odot(m_1\oplus m_2))}(x)<1-\theta \text{ and } \nu_{((r\odot m_1)\oplus(r\odot m_2))}(y)<1-\theta \text{ imply } x=y\end{cases}$

denote $((r_1 or r_2) \odot m)^{IF} = (\mu_{((r_1 or r_2)\odot m)}, \nu_{((r_1 or r_2)\odot m)})$, it implies

- $\begin{cases} \mu_{((r_1 or r_2)\odot m)}(x) > \theta \text{ and } \mu_{((r_1\odot m)\oplus(r_2\odot m))}(y) > \theta \text{ imply } x = y \\ \nu_{((r_1 or r_2)\odot m)}(x) < 1-\theta \text{ and } \nu_{((r_1\odot m)\oplus(r_2\odot m))}(y) < 1-\theta \text{ imply } x = y \end{cases}$

denote $((r_1 * r_2) \odot m)^{IF} = (\mu_{((r_1 * r_2)\odot m)}, \nu_{((r_1 * r_2)\odot m)})$, it implies

- $\begin{cases} \mu_{((r_1 * r_2)\odot m)}(x) > \theta \textit{ and } \mu_{(r_1\odot(r_2\odot m))}(y) > \theta \text{ imply } x = y \\ \nu_{((r_1 * r_2)\odot m)}(x) < 1-\theta \textit{ and } \nu_{(r_1\odot(r_2\odot m))}(y) < 1-\theta \text{ imply } x = y \end{cases}$

Proposition 2.17. *Let (R, G, H) be an IF ring and let (M, J) be an $R-$IFmodule; then for all $r, r_1, r_2 \in R, m, m_1, m_2 \in M$,*

1. $\begin{cases} \mu_{(r\odot(m_1\oplus m_2))}(x) > \theta \iff \mu_{((r\odot m_1)\oplus(r\odot m_2))}(x) < 1-\theta \\ \nu_{(r\odot(m_1\oplus m_2))}(x) < 1-\theta \iff \nu_{((r\odot m_1)\oplus(r\odot m_2))}(x) < 1-\theta \end{cases}$

2. $\begin{cases} \mu_{((r_1 or r_2)\odot m)}(x) > \theta \iff \mu_{((r_1\odot m)\oplus(r_2\odot m))}(x) > \theta \\ \nu_{((r_1 or r_2)\odot m)}(x) < 1-\theta \iff \nu_{((r_1\odot m)\oplus(r_2\odot m))}(x) < 1-\theta \end{cases}$

3. $\begin{cases} \mu_{((r_1 * r_2)\odot m)}(x) > \theta \iff \mu_{(r_1\odot(r_2\odot m))}(y) > \theta \\ \nu_{((r_1 * r_2)\odot m)}(x) < 1-\theta \iff \nu_{(r_1\odot(r_2\odot m))}(y) < 1-\theta \end{cases}$

Proof. It is clear by definitions. □

Remark 2.18. Let (G, R) be an IF group, then $(a \circ b)(d) > 0$ and $(a \circ c)(d) > 0$ imply $b = c$.

Proof. Let b be an inverse element of a, then with $((b \circ a) \circ a)^{IF} = (\mu_{((b\circ a)\circ a)}, \nu_{((b\circ a)\circ a)})$ and $(b \circ (a \circ a))^{IF} = (\mu_{(b\circ(a\circ a))},\ \nu_{(b\circ(a\circ a))})$, we have

$$\mu_{((b\circ a)\circ a)}(a) \geq \mu_R(b, a, e) \wedge \mu_R(e, a, a) > \theta,$$

$$\nu_{((b\circ a)\circ a)}(a) \leq \nu_R(b, a, e) \vee \nu_R(e, a, a) < 1-\theta,$$

$$\mu_{(b\circ(a\circ a))}(e) \geq \mu_R(a, a, a) \wedge \mu_R(b, a, e) > \theta,$$

$$\nu_{(b\circ(a\circ a))}(e) \leq \nu_R(a, a, a) \vee \nu_R(b, a, e) < 1-\theta.$$

It follows that $a = e$. □

Proposition 2.19. *Let (R, G, H) be an IF ring with zero element e_0 and (M, J) be a left $R-$fmodule with identity element e_j. Then for all $r \in R$, $m \in M$*

(1) $(r \odot e_j)^{IF} = (\mu_{(r\odot e_j)},\ \nu_{(r\odot e_j)}) \Rightarrow \begin{cases} \mu_{(r\odot e_j)}(e_j) > \theta \\ \nu_{(r\odot e_j)}(e_j) < 1-\theta \end{cases}$

(2) $(e_0 \odot m)^{IF} = (\mu_{(e_0\odot m)},\ \nu_{(e_0\odot m)}) \begin{cases} \mu_{(e_0\odot m)}(e_j) > \theta \\ \nu_{(e_0\odot m)}(e_j) < 1-\theta \end{cases}$

(3) $(r \odot m)^{IF} = (\mu_{(r\odot m)},\ \nu_{(r\odot m)}) \Rightarrow \begin{cases} \mu_{(r\odot m)}(x) > \theta \Longrightarrow \mu_{(r\odot m^{-1})}(x^{-1}) > \theta \\ \nu_{(r\odot m)}(x) < 1-\theta \Longrightarrow \nu_{(r\odot m^{-1})}(x^{-1}) < 1-\theta \end{cases}$

(4) $(r \odot m)^{IF} = (\mu_{(r\odot m)},\ \nu_{(r\odot m)}) \Rightarrow \begin{cases} \mu_{(r\odot m)}(x) > \theta \Longrightarrow \mu_{(r^{-1}\odot m)}(x^{-1}) > \theta \\ \nu_{(r\odot m)}(x) < 1-\theta \Longrightarrow \nu_{(r^{-1}\odot m)}(x^{-1}) < 1-\theta \end{cases}$

Proof.

(1) Let $x \in M$ such that $\begin{cases} \mu_{(r\odot e_j)}(e_j) > \theta \\ \nu_{(r\odot e_j)}(e_j) < 1-\theta \end{cases}$. then by $(r \odot (e_J \oplus e_J))^{IF} = (\mu_{(r\odot(e_J\oplus e_J))},\ \nu_{(r\odot(e_J\oplus e_J))})$

$$\begin{cases} \mu_{(r\odot(e_J\oplus e_J))}(x) > \mu_J(e_J,e_J,e_J) \wedge \mu_p(r,e_j,x) > \theta \\ \nu_{(r\odot(e_J\oplus e_J))}(x) < \nu_J(e_J,e_J,e_J) \vee \nu_p(r,e_j,x) < 1-\theta \end{cases}$$

it follows that $((r\odot e_j)\oplus(r\odot e_j))^{IF} = (\mu_{((r\odot e_j)\oplus(r\odot e_j))},\ \nu_{((r\odot e_j)\oplus(r\odot e_j))}) \Rightarrow \begin{cases} \mu_{((r\odot e_j)\oplus(r\odot e_j))}(x) > \theta \\ \nu_{((r\odot e_j)\oplus(r\odot e_j))}(x) < 1-\theta \end{cases}$ from proposition 2.17. then

$$\begin{cases} \mu_{((r\odot e_j)\oplus(r\odot e_j))}(x) > \mu_P(r,e_J,x) \wedge \mu_P(r,e_J,x) \wedge \mu_J(x,x,x) > \theta \\ \nu_{((r\odot e_j)\oplus(r\odot e_j))}(x) < \nu_P(r,e_J,x) \vee \nu_P(r,e_J,x) \vee \nu_J(x,x,x) < 1-\theta \end{cases}$$

Thus $\begin{cases} \mu_J(x,x,x) > \theta \\ \nu_J(x,x,x) < 1-\theta \end{cases}$ and $x = e_j$ from Remark 2.18.

(2) Let $x \in M$ such that $\begin{cases} \mu_{(e_0\odot m)}(e_j) > \theta \\ \nu_{(e_0\odot m)}(e_j) < 1-\theta \end{cases}$ then by $((e_0 \circ e_0)\odot m)^{IF} = (\mu_{((e_0\circ e_0)\odot m)},\ \nu_{((e_0\circ e_0)\odot m)})$

$$\begin{cases} \mu_{((e_0\circ e_0)\odot m)}(x) > \mu_G(e_0,e_0,e_0) \wedge \mu_p(e_0,m,x) > \theta \\ \nu_{((e_0\circ e_0)\odot m)}(x) < \nu_G(e_0,e_0,e_0) \vee \nu_p(e_0,m,x) < 1-\theta \end{cases}$$

It follows that
$((e_0\odot m)\oplus(e_0\odot m))^{IF} = (\mu_{((e_0\odot m)\oplus(e_0\odot m))},\ \nu_{((e_0\odot m)\oplus(e_0\odot m))}) \Rightarrow \begin{cases} \mu_{((e_0\odot m)\oplus(e_0\odot m))}(x) > \theta \\ \nu_{((e_0\odot m)\oplus(e_0\odot m))}(x) < 1-\theta \end{cases}$ from proposition 2.17. Then
$\begin{cases} \mu_{((e_0\odot m)\oplus(e_0\odot m))}(x) > \mu_p(e_0,m,x) \wedge \mu_P(e_0,m,x) \wedge \mu_J(x,x,x) > \theta \\ \nu_{((e_0\odot m)\oplus(e_0\odot m))}(x) < \nu_p(e_0,m,x) \vee \nu_P(e_0,m,x) \vee \nu_J(x,x,x) < 1-\theta \end{cases}$
Thus similar to (1), $\begin{cases} \mu_J(x,x,x) > \theta \\ \nu_J(x,x,x) < 1-\theta \end{cases}$ and so $x = e_J$.

(3) Let $\begin{cases} \mu_P(r,m,x) > \theta \\ \nu_P(r,m,x) < 1-\theta \end{cases}$ and let $y \in M$ such that $\begin{cases} \mu_P(r,m^{-1},y) > \theta \\ \nu_P(r,m^{-1},y) < 1-\theta \end{cases}$ $(r\odot(m\oplus m^{-1}))^{IF} = (\mu_{(r\odot(m\oplus m^{-1}))},\ \nu_{(r\odot(m\oplus m^{-1}))}) \Rightarrow$

$$\begin{cases} \mu_{(r\odot(m\oplus m^{-1}))}(e_J) > \mu_J(m,m^{-1},e_J) \wedge \mu_P(r,e_J,e_J) > \theta \\ \nu_{(r\odot(m\oplus m^{-1}))}(e_J) < \nu_J(m,m^{-1},e_J) \vee \nu_P(r,e_J,e_J) < 1-\theta \end{cases}$$

by proposition 2.17 we have $((r\odot m)\oplus(r\odot m^{-1}))^{IF} = (\mu_{((r\odot m)\oplus(r\odot m^{-1}))},\ \nu_{((r\odot m)\oplus(r\odot m^{-1}))})$ so

$$\begin{cases} \mu_{((r\odot m)\oplus(r\odot m^{-1}))}(e_J) > \theta \\ \nu_{((r\odot m)\oplus(r\odot m^{-1}))}(e_J) < 1-\theta \end{cases}$$

Therefore $\begin{cases} \mu_J(x,y,e_j) > \theta \\ \nu_J(x,y,e_j) < 1-\theta \end{cases}$ and consequently $y = x^{-1}$.

(4) It is obtain similar to (3). □

Proposition 2.20. *If (R,G,H) is a IF ring and K is any IF subring of R, then R is a $K-$IFmodule.*

Proof. Let (R,G,H) is a IF ring and let K is any IF subring of R Consider the mapping

$$\left\{P = (\mu_P,\nu_P) \mid \begin{array}{c} \mu_P : \mu_K \times \mu_R \longrightarrow \mu_R \\ \nu_P : \nu_K \times \nu_R \longrightarrow \nu_R \end{array}\right. ,$$

defined by $P(k,r) = H(k,r)$. It is obviously a IF function which satisfies the conditions in definitions 2.16. Moreover observe that (R,G) is necessarily an abelian IF group. consequently R is a left $K-$ifmodule. □

2.3 IF Submodule and IF Module Homomorphisms

Definition 2.21. Let (R, G, H) be an IF ring, (M, J) an $R-$IFmodule, and N a nonempty subset of M. If (N, J) is a $R-$IFmodule, N is called an *IF submodule* of M.

By definition we have the following Proposition trivially:

Proposition 2.22. *Let (R, G, H) be an IF ring, (M, J) a $R-$IFmodule, and N a nonempty subset of M. Then N is an IF submodule of M if and only if*

(1) (N, J) is an IF subgroup of (M, J);

(2) for all $r \in R$, $b \in N$, $(r \odot b)^{IF} = \begin{cases} \mu_{(r\odot b)}(c) > \theta \\ \nu_{(r\odot b)}(c) < 1-\theta \end{cases}$ implies $c \in N$.

Proposition 2.23. *If $\{N_i \mid i \in I\}$ is a family of IF submodules of an IF module M, then $\bigcap\limits_{i\in I} N_i$ is an IF submodule of M.*

Proof. It is clear. ☐

Definition 2.24. Let A and B be two IF modules over a IF ring (R, G, H) with a function $P : R\times M \longrightarrow M$. A function $f : A \longrightarrow B$ is an $R-$IFmodule homomorphism provided that, for all $a, b \in A$ and $r \in R$,

1. $\begin{cases} \mu_G(a, b, x) > \theta \\ \nu_G(a, b, x) < 1-\theta \end{cases}$ implies $\begin{cases} \mu_G(f(a), f(b), f(x)) > \theta \\ \nu_G(f(a), f(b), f(x)) < 1-\theta \end{cases}$;

2. $\begin{cases} \mu_P(r, a, x) > \theta \\ \nu_P(r, a, x) < 1-\theta \end{cases}$ implies $\begin{cases} \mu_P(r, f(a), f(x)) > \theta \\ \nu_P(r, f(a), f(x)) < 1-\theta \end{cases}$

Clearly, a $R-$IFmodule homomorphism $f : A \longrightarrow B$ is necessarily an abelian IF group homomorphism. Consequently the same terminology is used for IF modules: F is a $R-$IFmodule monomorphism (resp., epimorphism, isomorphism) if it is injective (resp., surjective, bijective) as IF group homomorphisms.

Let $f : A \longrightarrow B$ be an $R-$IFmodule homomorphism.Then the kernel and the image of f as IF group homomorphisms are denoted by

$$Kerf = \{a \in A \mid f(a) = e_B\},$$
$$Imf = \{b \in B \mid b = f(a), a \in A\},$$

respectively.

Theorem 2.25. *Let (R, G, H) be an IF ring and let $f : A \longrightarrow B$ be a $R-$IFmodule homomorphism.Then*

(1) f is an $R-$IFmodule monomorphism if and only if $Kerf = \{e_A\}$

(2) $f : A \longrightarrow B$ is an $R-$IFmodule isomorphism if and only if there exists an IF module homomorphism $G : B \longrightarrow A$ such that $gf = e_A$ and $fg = e_B$.

Theorem 2.26. *Let $f : (G_1, R_1) \longrightarrow (G_2, R_2)$ be an IF group homomorphism, then if H_2 is a fuzzy subgroup of G_2, then $f^{-1}(H_2)$ is a fuzzy subgroup of G_1.*

Proposition 2.27. *Let $f : A \longrightarrow B$ be a $R-$IFmodule homomorphism. Then*

1. $Kerf$ is an IF submodule of A;

2. Imf is an IF submodule of B;

3. if C is any IF submodule of B, then $f^{-1}(C) = \{a \in A | f(a) \in C\}$ is an IF submodule of A.

Proof.

(1) $Kerf$ is a fuzzy subgroup of the abelian fuzzy group A from Theorem 26 in [1] . Let $r \in R$ and $a \in Kerf$ such that $\left\{ \begin{array}{c} \mu_P(r,a,x) > \theta, \\ \nu_P(r,a,x) < 1-\theta \end{array} \right.$ Since f is a $R-$IFmodule homomorphism, $\left\{ \begin{array}{c} \mu_P(r,f(a),f(x)) > \theta, \\ \nu_P(r,f(a),f(x)) < 1-\theta \end{array} \right.$. On the other hand, as $a \in Kerf$ we have $f(a) = e_B$. Therefore $\left\{ \begin{array}{c} \mu_P(r,e_B,f(x)) > \theta, \\ \nu_P(r,e_B,f(x)) < 1-\theta \end{array} \right.$ and so $f(x) = e_B$ from Proposition 2.19. So $x \in Kerf$ is obtained.

(2) Imf is a IF subgroup of the abelian IF group A from Theorem 26 in [1]. For any $r \in R$, $b \in Imf$ there exists $a \in A$ such that $b = f(a)$. Let $x \in A$, $H = (\mu_H, \nu_H)$ such that $\left\{ \begin{array}{c} \mu_H(r,\ a,\ x) > \theta \\ \nu_H(r,\ a,\ x) < 1-\theta \end{array} \right.$. Since f is an $R-$ifmodule homomorphism, $\left\{ \begin{array}{c} \mu_H(r,\ f(A),\ f(x)) > \theta \\ \nu_H(r,\ f(A),\ f(x)) < 1-\theta \end{array} \right.$ which means $\left\{ \begin{array}{c} \mu_H(r,\ b,\ f(x)) > \theta \\ \nu_H(r,\ b,\ f(x)) < 1-\theta \end{array} \right.$ and so $f(x) \in B$.

(3) $f^{-1}(C)$ is an IF subgroup of the abelian IF group A from 2.26. Let $r \in R$ and $x \in f^{-1}(C)$ such that $\left\{ \begin{array}{c} \mu_H(r,\ x,\ u) > \theta \\ \nu_H(r,\ x,\ u) < 1-\theta \end{array} \right.$ Since $\left\{ \begin{array}{c} \mu_H(r,\ f(x),\ f(u)) > \theta \\ \nu_H(r,\ f(x),\ f(u)) < 1-\theta \end{array} \right.$ and $f(x) \in C$ we have that $f(u) \in C$ and $u \in f^{-1}(C)$. This completes the proof. $\square$

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