New Extension of a well-known method to solve a systems of two-dimensional Partial differential equations

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ABSTRACT

During past decades, various techniques were used to achieve the exact solution of the differential equation. But most of them have not been applied directly to manipulate the system of equations. In the current paper, a well-known and influential method is expanded to obtain some kind of solution of a two-dimensional system, without reducing of equations. As an example, the proposed method has been applied to achieve the exact solutions of a coupled dimensional Burgers equations. The graphs of the obtained results have been illustrated.

KEYWORDS: modified coupled Exp-function method; exact solution; Burgers equation.

1 INTRODUCTION

Finding analytical solutions of evolution equations has a vital role in various sciences because many physical phenomena can be explained by analysing them and that is why many researchers are interested in working on this topic in recent years. In the last decades, various methods have been introduced and used for this purpose [1-3].

In almost all of these methods, to solve a system of equations, first, it will be converted into an equation and then the desired method will be applied. The motivation of the present paper is to develop a widely used method called the modified exp function method to directly solve a system of differential equations.

The modified Exp-function method has been developed to attain different sort of solutions of nonlinear equations. This method was first proposed by He and Wu in 2006, and modifications have been used to solve many nonlinear equations so far.

In the present research, the above-mentioned method have been generalized to solve the system of differential equations. Especially the system two-dimensional Burgers equations (STDBE) as the following form, have been solved by them.

$$\begin{cases} u_t - 2uu_x - u_{xx} - u_{yy} - 2vu_y = 0, \\ v_t - 2vv_x - v_{xx} - v_{yy} - 2vv_y = 0. \end{cases} \tag{1}$$

2 COUPLED MODIFIED EXP FUNCTION METHOD

To explain the method, the following nonlinear system of PDE should be noted:

$$\begin{cases} F\left(u, \mathrm{v}, u_x, u_t, v_x, v_t, \ldots\right) = 0, \\ G\left(u, \mathrm{v}, u_x, u_t, v_x, v_t, \ldots\right) = 0. \end{cases} \tag{2}$$

By using the nonlinear complex transformation

$$\xi = \delta x - \rho t, \tag{3}$$

Where ρ and δ are nonzero parameters, Eq. (2) turns to a system of ODE

$$\begin{cases} f\left(u, \mathrm{v}, u', v', \ldots\right) = 0, \\ g\left(u, \mathrm{v}, u', v', \ldots\right) = 0. \end{cases} \tag{4}$$

In the Coupled modified exp function method (CMEF) the solution of system (4) have been imaged as:

$$u(\xi) = \sum_{i=-M}^{M} a_i (\exp(\varphi(\xi)))^i,\ v(\xi) = \sum_{i=-N}^{N} b_i (\exp(\varphi(\xi)))^i, \tag{5}$$

where $\varphi(\xi)$ satisfies the nonlinear ODE in the form as fallows

$$\varphi' = \exp(-\varphi) + \alpha \exp(\varphi) + \beta \tag{6}$$

$a_i, \mathrm{b}_i, \alpha,$ and β are parameters to be handled further. The solution of Eq. (6) are as follows

$$\begin{aligned}
&1. If\ \alpha \neq 0,\ \beta^2 - 4\alpha > 0,\ \ \varphi(\xi) = \ln\left(-\frac{\sqrt{\beta^2 - 4\alpha}}{2\alpha} \tanh\left(\frac{\sqrt{\beta^2 - 4\alpha}}{2} (\xi + c) \right) - \frac{\beta}{2\alpha} \right), \\
&2. If\ \alpha \neq 0,\ \beta^2 - 4\alpha < 0,\ \ \varphi(\xi) = \ln\left(\frac{\sqrt{4\alpha - \beta^2}}{2\alpha} \tan\left(\frac{\sqrt{4\alpha - \beta^2}}{2} (\xi + c) \right) - \frac{\beta}{2\alpha} \right), \\
&3. If\ \alpha \neq 0,\ \beta^2 - 4\alpha = 0,\ \ \varphi(\xi) = \ln\left(-\frac{2\beta(\xi + c) + 4}{\beta(\xi + c)} \right), \\
&4.\ If\ \alpha = 0,\ \beta \neq 0,\ \ \varphi(\xi) = -\ln\left(\frac{\beta}{\exp(\beta(\xi + c)) - 1} \right), \\
&5.\ If\ \alpha = 0,\ \beta = 0,\ \ \varphi(\xi) = \ln\left(\xi + c\right),
\end{aligned} \tag{7}$$

To obtain the numbers M and N, we strike a balance between the sentences with the topmost derivative and the topmost nonlinear order in EQ. (4). Placing Eq. (6) into Eq. (4) and considering Eq. (7) precede an algebraic system including powers of $\exp(\varphi(\xi))$. By setting the coefficient of these power to zero, $a_i, b_i, \tau, \upsilon, \alpha,$ and β are acquire. Finally substituting over determine value in eq. (6), and the general solutions of (7), the exact solutions of the Eq. (2) will be achieved.

3 APPLICATION MODIFIED EXP-FUNCTION METHOD

At first, we introduce a complex variable ξ, defined as

$$\xi = \delta x - \rho t + \lambda y,$$

So, Eq. (1) turns to the following system of ordinary different equation,

$$\begin{cases} \rho U' + 2\delta U U' + (\delta^2 + \lambda^2) U'' + 2\lambda V U' = 0, \\ \rho V' + 2\delta U V' + (\delta^2 + \lambda^2) V'' + 2\lambda V V' = 0. \end{cases} \tag{8}$$

The solution of (13) will be image as Eq.(5), Where $\varphi(\xi)$ satisfy in Eq. (6). Homogeneous balance between linear and nonlinear terms in each equation of (8) leads to $M=1,$ and $N=1$. So the solution (5), will be written as follows

$$\begin{cases} u(\xi)=a_1 e^{\varphi(\xi)}+a_0+a_{-1}e^{-\varphi(\xi)}, \\ v(\xi)=b_1 e^{\varphi(\xi)}+b_0+b_{-1}e^{-\varphi(\xi)}. \end{cases} \quad (9)$$

Putting (9) in (8), and putting the coefficient of $e^{\varphi(\xi)}$ equal to zero, yields a system of algebraic equations, Which solving by maple leads to

Case 1: $a_1=0,\ b_{-1}=\dfrac{\delta^2-\delta a_{-1}+\lambda^2}{\lambda},\ b_0=\dfrac{1}{2}\dfrac{\beta\delta^2-2\delta a_0+\beta\lambda^2-\rho}{\lambda}, b_1=0,$ (10)

By placing above solution into (10), the following exact solution will be derived

$$\begin{cases} u(\xi)=a_0+a_{-1}e^{-\varphi(\xi)}, \\ v(\xi)=\dfrac{1}{2}\dfrac{\beta\delta^2-2\delta a_0+\beta\lambda^2-\rho}{\lambda}+\dfrac{\delta^2-\delta a_{-1}+\lambda^2}{\lambda}e^{-\varphi(\xi)}. \end{cases} \quad (11)$$

If $\alpha\neq 0,\ \beta^2-4\alpha>0,$

$$\begin{cases} u(\xi)=a_0+\dfrac{2\alpha a_{-1}}{\left(-\sqrt{\beta^2-4\alpha}\tanh\left(\dfrac{\sqrt{\beta^2-4\alpha}}{2}(\xi+c)\right)-\beta\right)}, \\ v(\xi)=\dfrac{1}{2\lambda}(\beta\delta^2-2\delta a_0+\beta\lambda^2-\rho)+\dfrac{2\alpha(\delta^2-\delta a_{-1}+\lambda^2)}{\left(-\sqrt{\beta^2-4\alpha}\tanh\left(\dfrac{\sqrt{\beta^2-4\alpha}}{2}(\xi+c)\right)-\beta\right)}. \end{cases} \quad (12)$$

If $\alpha\neq 0,\ \beta^2-4\alpha<0,$

$$\begin{cases} u(\xi)=a_0+\dfrac{2\alpha a_{-1}}{\sqrt{4\alpha-\beta^2}\tan\left(\dfrac{\sqrt{4\alpha-\beta^2}}{2}(\xi+c)\right)-\beta}, \\ v(\xi)=\dfrac{1}{2\lambda}(\beta\delta^2-2\delta a_0+\beta\lambda^2-\rho)+\dfrac{2\alpha(\delta^2-\delta a_{-1}+\lambda^2)}{\sqrt{4\alpha-\beta^2}\tan\left(\dfrac{\sqrt{4\alpha-\beta^2}}{2}(\xi+c)\right)-\beta}. \end{cases} \quad (13)$$

If $\alpha\neq 0,\ \beta^2-4\alpha=0,$

$$\begin{cases} u(\xi)=a_0-\dfrac{a_{-1}\beta(\xi+c)}{2\beta(\xi+c)+4}, \\ v(\xi)=\dfrac{1}{2}\dfrac{\beta\delta^2-2\delta a_0+\beta\lambda^2-\rho}{\lambda}-\dfrac{\beta(\xi+c)(\delta^2-\delta a_{-1}+\lambda^2)}{\lambda(2\beta(\xi+c)+4)}. \end{cases} \quad (14)$$

If $\alpha=0,\ \beta\neq 0,$

$$\begin{cases} u(\xi)=a_0+\dfrac{\beta a_{-1}}{\exp(\beta(\xi+c))-1}, \\ v(\xi)=\dfrac{1}{2\lambda}(\beta\delta^2-2\delta a_0+\beta\lambda^2-\rho)+\dfrac{\beta(\delta^2-\delta a_{-1}+\lambda^2)}{\exp(\beta(\xi+c))-1}. \end{cases} \quad (15)$$

If $\alpha = 0,\ \beta = 0,$

$$\begin{cases} u(\xi) = a_0 + \dfrac{a_{-1}}{(\xi + c)}, \\ v(\xi) = \dfrac{1}{2}\dfrac{-2\delta a_0 - \rho}{\lambda} + \dfrac{\delta^2 - \delta a_{-1} + \lambda^2}{\lambda(\xi + c)}. \end{cases} \tag{16}$$

In Fig. 1-4 the plots of solutions (11) - (16), for some values of parameters, have been illustrated.

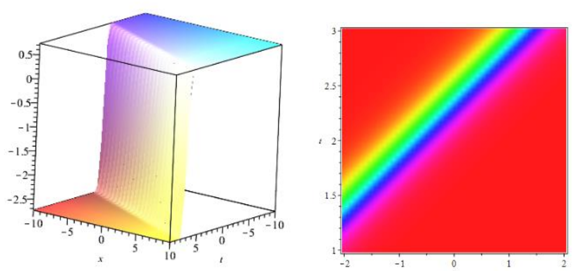


Fig.1 : plots of *solution* (17) *for* $u(x,1,t), \alpha = 1, \beta = 4, a_{-1}=\delta=c=y=1, \lambda=3$ and $\rho=2$.

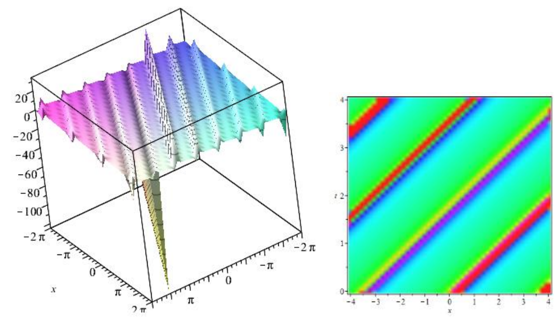


Fig.2 : plots of *solution* (18) *for* $u(x,1,t), \alpha = 1, \beta = 1, a_{-1}=\delta=c=y=1, \lambda=3$ and $\rho=2$.

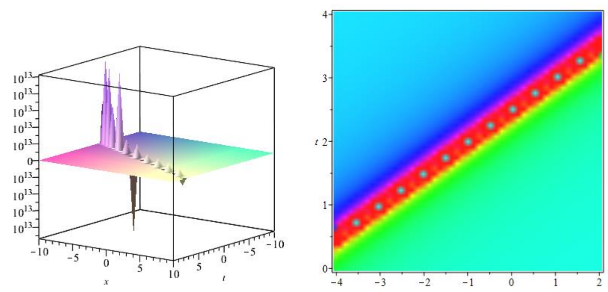


Fig.3 : plots of *solution* (19) *for* $u(x,1,t), \alpha = 1, \beta = 2, a_{-1}=\delta=c=y=1, \lambda=3$ and $\rho=2$.

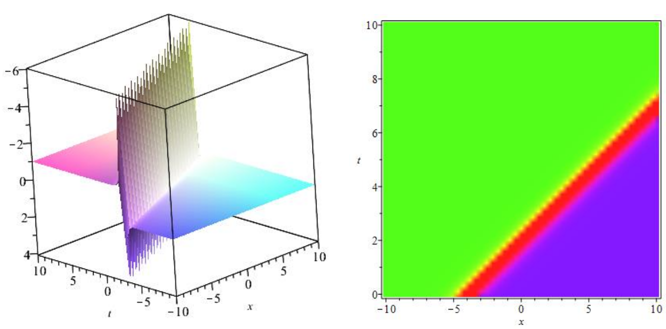


Fig.4 : plots of *solution* (20) *for* $u(x,1,t), \beta = 2, a_{-1}=\delta=c=y=1, \lambda=3$ and $\rho=2$.

Case 2:

$$\alpha=0,\lambda=-\frac{\delta a_1}{b_1},\ a_{-1}=0,\ b_{-1}=0,\ \rho=\frac{\delta(\beta\delta a_1^2+\beta\delta b_1^2+2a_0b_1^2-2a_1b_0b_1}{b_1^2}.$$

By placing above solution into (9), the following exact solution will be derived

$If\ \ \beta\neq 0,$

$$\begin{cases} u(\xi)=\dfrac{a_1}{\beta}(\exp(\beta(\delta x-\dfrac{\delta(\beta\delta a_1^2+\beta\delta b_1^2+2a_0b_1^2-2a_1b_0b_1}{b_1^2}t-\dfrac{\delta a_1}{b_1}y+c))-1)+a_0, \\ v(\xi)=\dfrac{b_1}{\beta}(\exp(\beta(\delta x-\dfrac{\delta(\beta\delta a_1^2+\beta\delta b_1^2+2a_0b_1^2-2a_1b_0b_1}{b_1^2}t-\dfrac{\delta a_1}{b_1}y+c))-1)+b_0. \end{cases} \quad (17)$$

$If\ \ \beta=0,$

$$\begin{cases} u(\xi)=a_1\left(\delta x-\dfrac{\delta(2a_0b_1^2-2a_1b_0b_1}{b_1^2}t-\dfrac{\delta a_1}{b_1}y+c\right)+a_0, \\ v(\xi)=b_1\left(\delta x-\dfrac{\delta(2a_0b_1^2-2a_1b_0b_1}{b_1^2}t-\dfrac{\delta a_1}{b_1}y+c\right)+b_0. \end{cases} \quad (18)$$

In Fig. 5 the plot of solution (17) for some values of parameters, has been illustrated.

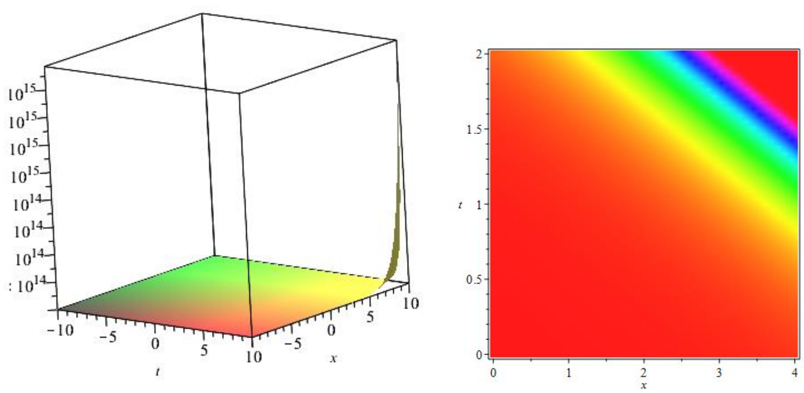


Fig.5 : plots of *solution* (22) *for* $u(x,1,t)$, $\beta=1, a_0=a_1=b_0=1$, $\delta=1$ and $b_1=3$.

case 3:

$$\rho=-\beta\delta^2-\beta\lambda^2-2\delta a_0-2\lambda b_0,\ a_{-1}=0,\ b_{-1}=0,\ a_1=-\frac{\alpha\delta^2+\alpha\lambda^2+\lambda b_1}{\delta}.$$

By placing above solution into (14), the following exact solution will be derived

If $\alpha \neq 0,\ \beta^2 - 4\alpha > 0,$

$$\begin{cases} u(\xi) = -\dfrac{\alpha\delta^2 + \alpha\lambda^2 + \lambda b_1}{\delta}\left(-\dfrac{\sqrt{\beta^2 - 4\alpha}}{2\alpha}\tanh\left(\dfrac{\sqrt{\beta^2 - 4\alpha}}{2}(\xi + c) \right) - \dfrac{\beta}{2\alpha} \right) + a_0, \\ v(\xi) = b_1\left(-\dfrac{\sqrt{\beta^2 - 4\alpha}}{2\alpha}\tanh\left(\dfrac{\sqrt{\beta^2 - 4\alpha}}{2}(\xi + c) \right) - \dfrac{\beta}{2\alpha} \right) + b_0. \end{cases} \tag{19}$$

If $\alpha \neq 0,\ \beta^2 - 4\alpha < 0,$

$$\begin{cases} u(\xi) = -\dfrac{\alpha\delta^2 + \alpha\lambda^2 + \lambda b_1}{\delta}\left(\dfrac{\sqrt{4\alpha - \beta^2}}{2\alpha}\tan\left(\dfrac{\sqrt{4\alpha - \beta^2}}{2}(\xi + c) \right) - \dfrac{\beta}{2\alpha} \right) + a_0, \\ v(\xi) = b_1\left(\dfrac{\sqrt{4\alpha - \beta^2}}{2\alpha}\tan\left(\dfrac{\sqrt{4\alpha - \beta^2}}{2}(\xi + c) \right) - \dfrac{\beta}{2\alpha} \right) + b_0. \end{cases} \tag{20}$$

If $\alpha \neq 0,\ \beta^2 - 4\alpha = 0,$

$$\begin{cases} u(\xi) = -\dfrac{\alpha\delta^2 + \alpha\lambda^2 + \lambda b_1}{\delta}\left(-\dfrac{2\beta(\xi + c) + 4}{\beta(\xi + c)} \right) + a_0, \\ v(\xi) = b_1\left(-\dfrac{2\beta(\xi + c) + 4}{\beta(\xi + c)} \right) + b_0. \end{cases} \tag{21}$$

If $\alpha = 0,\ \beta \neq 0,$

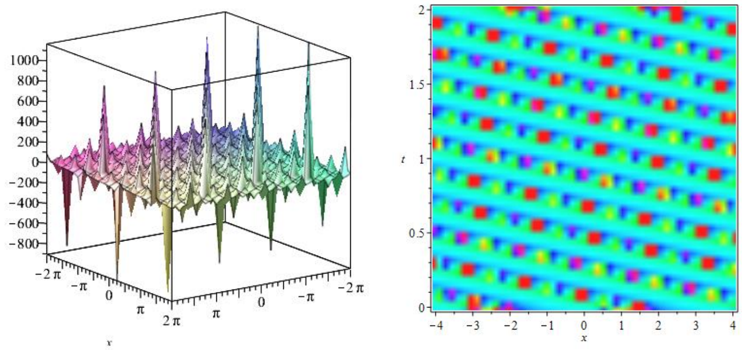
$$\begin{cases} u(\xi) = -\dfrac{\lambda b_1(\exp(\beta(\xi + c)) - 1)}{\delta\beta} + a_0, \\ v(\xi) = \dfrac{b_1(\exp(\beta(\xi + c)) - 1)}{\beta} + b_0. \end{cases} \tag{22}$$

where $\xi = \delta x + (\beta\delta^2 + \beta\lambda^2 + 2\delta a_0 + 2\lambda b_0)t + \lambda y,$.

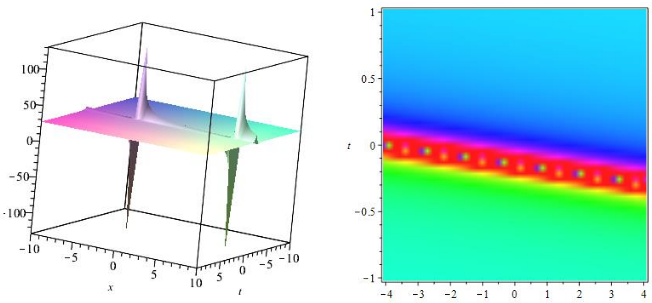
If $\alpha = 0,\ \beta = 0,$

$$\begin{cases} u(\xi) = -\dfrac{\lambda b_1}{\delta}(\xi + c) + a_0, \\ v(\xi) = b_1(\xi + c) + b_0. \end{cases} \tag{23}$$

where $\xi = \delta x + (2\delta a_0 + 2\lambda b_0)t + \lambda y$. In Fig. 6-7 the plots of solutions (20) and (21), for some values of parameters, have been illustrated.



$Fig.6$: plots of $solution$ (25) $for\ u(x,1,t),\ \beta=1, \alpha=1, a_0=b_1=b_0=1,\ \delta=1 \text{and}\ \lambda=3.$



$Fig.7$: plots of $solution$ (26) $for\ u(x,1,t),\ \beta=2, \alpha=1, a_0=b_1=b_0=1,\ \delta=1 \text{and}\ \lambda=3.$

case 4 :

$$\beta=0, \rho=-2\delta a_0-2\lambda b_0,\ \ a_{-1}=\frac{\alpha\delta^2+\alpha\lambda^2+\lambda b_1}{\alpha\delta}, a_1=-\frac{\alpha\delta^2+\alpha\lambda^2+\lambda b_1}{\delta}, b_{-1}=-\frac{b_1}{\alpha}$$

By placing above solution into (9), we derive

If $\alpha<0,$

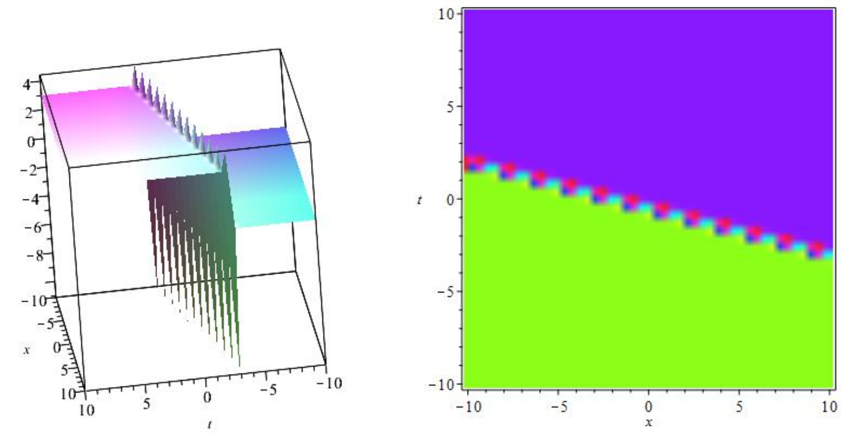
$$\begin{cases} u(\xi)=\dfrac{(\alpha\delta^2+\alpha\lambda^2+\lambda b_1)\sqrt{-\alpha}}{\alpha\delta}\tanh\left(\sqrt{-\alpha}(\xi+c)\right)+a_0-\dfrac{\alpha\delta^2+\alpha\lambda^2+\lambda b_1}{\delta\sqrt{-\alpha}\tanh\left(\sqrt{-\alpha}(\xi+c)\right)}, \\ v(\xi)=-\dfrac{b_1\sqrt{-\alpha}}{\alpha}\tanh\left(\sqrt{-\alpha}(\xi+c)\right)+b_0+\dfrac{b_1}{\sqrt{-\alpha}\tanh\left(\sqrt{-\alpha}(\xi+c)\right)}. \end{cases} \tag{24}$$

where $\xi=\delta x+(2\delta a_0+2\lambda b_0)t+\lambda y$.

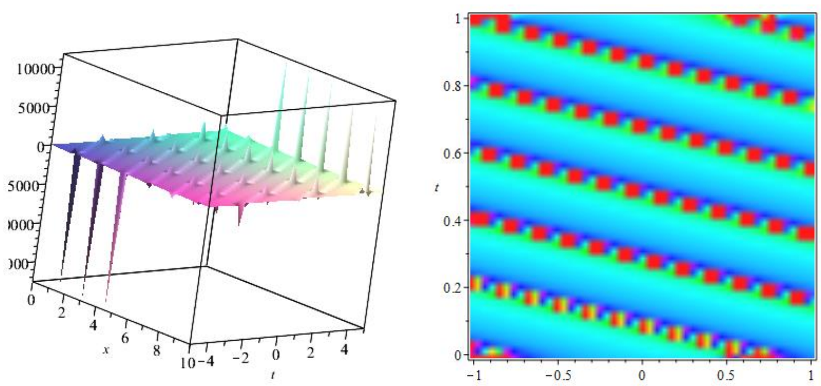
If $\alpha>0,$

$$\begin{cases} u(\xi)=-\dfrac{\alpha\delta^2+\alpha\lambda^2+\lambda b_1}{\delta}\dfrac{\sqrt{\alpha}}{\alpha}\tan\left(\sqrt{\alpha}(\xi+c)\right)+a_0+\dfrac{\alpha\delta^2+\alpha\lambda^2+\lambda b_1}{\delta\sqrt{\alpha}\tan\left(\sqrt{\alpha}(\xi+c)\right)}, \\ v(\xi)=b_1\dfrac{\sqrt{\alpha}}{\alpha}\tan\left(\sqrt{\alpha}(\xi+c)\right)+b_0-\dfrac{b_1}{\sqrt{\alpha}\tan\left(\sqrt{\alpha}(\xi+c)\right)}. \end{cases} \tag{25}$$

where $\xi = \delta x + (2\delta a_0 + 2\lambda b_0)t + \lambda y$. In Fig. 8-9 the plots of solutions (24) and (25), for some values of parameters, have been illustrated.



$Fig.8$: plots of $solution$ (29) $for\ \alpha = -1, \beta = 0, a_0 = b_1 = b_0 = \delta = \lambda = c = 1.$



$Fig.9$: plots of $solution$ (29) $for\ \alpha = 1, \beta = 0, a_0 = b_1 = b_0 = \delta = \lambda = c = 1.$

4 CONCLUSION

In this paper, new modifications of Exp function method has been expanded to attain the generalized solutions of a system of PDE. Implementing the methods for solving systems of PDE directly is an innovative attempt. So, all of the system where cannot reduce to an equation is solvable by these methods comfortably. The system of two-dimensional Burgers equations (STDBE) have been resolved by above-mentioned method and several kinds of solutions of the above equation have been achieved. The results indicate that the technique is useful means to get the exact solutions of the system of PDE.

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