8th International Conference on
Combinatorics, Cryptography, Computer Science and Computation
November. 15-16, 2023

Cycle basis of graphs

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Abstract

Cycle in graph is an important structure of graph which appear in many areas of mathematics, science and engineering. For instance cycles paly important rule in periodic scheduling, graph drawing, analysis of chemical and biological pathways and , analysis of electrical networks. Cycle space of graph is a linear space which contains all cycles of graphs and also all linear combination of cycles, which is considered over different fields. A cycle basis is a basis of cycle space. In fact a cycle basis is a compact representation of the set of all cycles in graph. There are different types of cycle basis. More precisely strictly fundamental, weakly fundamental, totally unimodular, integral, directed and undirected basis are some kinds of cycle basis of graphs. In this paper we we consider some kinds of cycle basis for special family of graphs.

Keywords: Cycles in graphs, Cycle basis, Minimum cycle basis.
AMS Mathematical Subject Classification [2010]: 05C50, 05C90

1 Introduction

A cycle of a graph $G = (V, E)$ is a connected regular subgraph of degree 2. Any cycle in G can be represented by an incidence vector $\gamma_C \in \{0, 1\}^{|E|}$ ($\gamma_C \in \{0, \pm 1\}^{|E|}$ in directed case). The *cycle space* of G is the vector space generated by $\{\gamma_C| \ \ C \ \text{ is a cycle in } \ G\}$ over $\mathbb{Z}_2$ (over $\mathbb{Q}$ in directed case). A *cycle basis* for G consists of some cycles which form a basis for cycle space of G. The *length of a cycle basis* is the total length of the cycles included in the basis. A *minimum cycle basis* (or MCB for short) of a graph is a cycle basis with minimum length. In [5] the authors give a good survey on cycle basis of graphs. In [7] five different classes of cycle bases are defined. Here we give this definition from [7].

Definition 1.1. Let $B = \{C_1, C_2, \ldots, C_\nu\}$ be a directed cycle basis for a graph G, then we have following definitions:

- **Integral basis**; B is called integral if every cycle C of G can be written as an integral combination of the members of B, i. e.
$$\exists \lambda_i \in \mathbb{Z}; C = \lambda_1 C_1 + \lambda_2 C_2 + \ldots + \lambda_\nu C_\nu.$$
- **Totally unimodular basis**; B is called totally unimodular if every cycle of G can be written as a linear combination of members of B with coefficient in $\{0, \pm 1\}$

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- Weakly fundamental basis; if there exits some permutation σ such that

$$C_{\sigma(i)} \setminus C_{\sigma(1)} \cup \ldots \cup C_{\sigma(i-1)} \neq \emptyset, \forall i = 2, \ldots, \nu; \tag{1}$$

- **Strictly fundamental basis**; B is called fundamental if there exits some spanning tree T of G such that $\mathcal{B} = \{C_T(e) | e \in E(G) \setminus E(T)\}$, where $C_T(e)$ denoted the unique cycle in $T \cup \{e\}$.
- **Planar, or 2-basis**; B is called planar basis if each arc is contained in at most two basic circuits and the basis is undirected.

It is easy to see that every fundamental basis is weakly fundamental and every weakly fundamental is integral basis. But finding minimum basis in each class is an interesting problem. The problem of finding minimum weakly fundamental basis is an APX-hard problem. Hence, solving this problem for special family of graphs is interesting. In this paper we find a minimum cycle basis for some special graph products.

2 Minimum Weakly Fundamental Basis

Let $K_{n,n}$ be a complete bipartite graph of order $2n$ and M be a complete matching of $K_{n,n}$. Suppose G_n be a graph constructed from $K_{n,n} \setminus M$. Then for each n we will find a minimum weakly fundamental basis for graph G_n. The proof is mostly based on work in [3].

Lemma 2.1. *For any integer $n \geq 3$, graph G_n has a fundamental cycle basis which consists of one cycle of length 6 and all other cycles of length 4.*

Proof. Let T be a spanning tree of G_n with edge set $E(T) = \{\{1, i\} : n+2 \leq i \leq 2n\} \cup \{\{j, n+2\} : 3 \leq j \leq n\} \cup \{\{3, n+1\}, \{2, n+3\}\}$. Let $\mathcal{B}$ be the fundamental basis with respect to T. We claim that $\mathcal{B}$ has the desired property. To prove the claim, let $e = \{r, s\}$ be a non-tree edge of G_n with $r \in A$ and $s \in B$. There are four possibilities:
Case 1. $r \neq 2$ and $s \neq n+1$. In this case the cycle $C_T(e) = (r, s, 1, n+2)$.
Case 2. $r = 2$ and $s \neq n+1$. Then the cycle $C_T(e) = (2, s, 1, n+3)$.
Case 3. $r \neq 2$ and $s = n+1$. Then the cycle $C_T(e) = (r, n+1, 3, n+2)$.
Case 4. $r = 2$ and $s = n+1$. Then the cycle $C_T(e) = (2, n+1, 3, n+2, 1, n+3)$.
Hence, $\mathcal{B}$ contains one cycle of length 6 and $n^2 - 3n$ cycles of length 4. □

Theorem 2.2. *For every $n \geq 4$, graph G_n has a weakly fundamental cycle basis all whose elements are cycles of length 4.*

Proof. Let $\mathcal{B}$ be the fundamental cycle basis for G_n and T be the spanning tree for G_n, constructed in the proof of Lemma 2.1. Let $\mathcal{B}' = \mathcal{B} \setminus C_T(\{2, n+1\}) \cup \{C(\{2, n+1\})\}$, where $C(\{2, n+1\}) = (2, n+1, 3, n+4)$. If σ be any permutation on the elements of $\mathcal{B}'$ with $\sigma(n^2 - 3n + 1) = C(\{2, n+1\})$, then the elements of $\mathcal{B}'$ satisfy (1). Hence $\mathcal{B}'$ is weakly fundamental cycle basis. □

Corollary 2.3. *For any integer $n \geq 4$, minimum weakly fundamental, integral and undirected cycle bases for G_n contain cycles of length 4.*

Proof. all cycles of G_n have length at least 4, so the cycle basis $\mathcal{B}'$, introduced in the proof of Theorem 2.2 is minimum weakly fundamental cycle basis. Using Corollary 30 and Corollary 28 of [7] $\mathcal{B}'$ is also minimum integral and minimum undirected cycle basis. □

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