A Probabilistic Approach to the Derangement Problem

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ABSTRACT

Assume that $S_n(\mathrm{r}) \subseteq S_n$. Also consider that $S_n(\mathrm{r})$ is the set of permutations of size n with r fixed points. Then:

$$|S_n(\mathrm{r})| = \binom{n}{\mathrm{r}} D_{n-\mathrm{r}}$$

Where, D_n is the number equal to the number of members of a derangement set of size n. Assume that p is a real number such that $0 \leq p < 1$. Also, consider that $1 \leq r < n$. If the probability that a permutation with n elements has at least r fixed members is less than the number p, then the number r is independent of the choice of the number n. In addition,

$$\left(\frac{1}{0!} + \cdots + \frac{1}{r-1!}\right) > (1-\mathrm{p})\mathrm{e}$$

KEYWORDS: Derangement, Combinatorial method, Probabilistic Approach, Permutation.

1 INTRODUCTION

Derangements are arrangements of some number of objects into positions such that no object goes to its specified position. In the language of permutations, a derangement is a permutation in which none of the objects appear in their "natural" (i.e., ordered) place.

If we choose a random permutation, the probability that it is a derangement is close to 1/e. Another version of the problem arises when we ask for the number of ways n letters, each addressed to a different person, can be placed in n pre-addressed envelopes so that no letter appears in the correctly addressed envelope.

The derangement problem was formulated by P. R. de Montmort in 1708, and solved by him in 1713 (de Montmort 1713-1714). Nicholas Bernoulli also solved the problem using the inclusion-exclusion principle. The number of derangements of an n-element set is called the n-th derangement number or rencontres number, or the sub-factorial of n and is sometimes denoted D_n. Counting the derangements of a set amounts to what is known as the hat-check problem, in which one considers the number of ways in which n hats can be returned to n people such that no hat makes it back to its owner. This number satisfies the recurrences

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Also, it is well-known that

$$\lim_{n\to\infty}\frac{D_n}{n!}=e^{-1}=0.3678\ldots$$

THEOREM A. Assume that A is a subset of $\{1,2,3,\ldots,n\}$ and consider $\sigma\boldsymbol{\epsilon S_n}$is a derangement on A. Also assume that $|A|=m\le n$ and D_m is the set of all derangement on A, then:

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{n-k}{m-k}(m-k)!$$

Theorem B. For all $n>2$ and $r>0$, we have

$$D_r(n)=rD_{r-1}(n-1)+(n-1)D_r(n-2)+(n+r-1)D_r(n-1)$$

Also, they find exponential generating function of $D_r(n)$ as following:

Theorem C. For any $r\epsilon\mathbb{N}$ for the exponential generating function of the sequence of r-derangements numbers we have that:

In this manner, we obtained the following information in previous works.

$$F_r(x)=\sum_{n=0}^{\infty}\frac{D_r(n)}{n!}x^n=\frac{x^r e^{-x}}{(1-x)^{r+1}}$$

The interesting this is that the number e itself also has applications in probability theory, in a way that is not obviously related to exponential growth. Suppose that a gambler plays a slot machine that pays out with a probability of one in n and plays it n times. Gordon and McMahon noted that the number of derangements in the hyperoctahedral group gives the rising 2-binomial transform of the derangement numbers for S_n. More generally, they shows that the cyclic derangement numbers give a mixed version of the rising r-binomial transform and falling $(r-1)$ binomial transform of D_n. This new hybrid k-binomial transform may share many of the nice properties of Spivey and Steil's transforms, including Hankel invariance and/or a simple description of the change in the exponential generating function. Further, it could be interesting to evaluate the expression for negative or even non-integer values of k. For instance, taking $k=1/2$ gives the binomial mean transform which is of some interest.

We define a new special case of derangement, and also we obtain some relation on this subset of derangements. This special case of derangement is a subset of block derangement.

Assume that $\sigma\epsilon S_n$ is a permutation on n elements, for example $\{1,2,3,\ldots,n\}$. Consider that k is an integer such that $0\le k<n$. We define

$$X_k=\{\sigma\epsilon S_n\ |\sigma(i+1)+k\neq\sigma(i)\ for\ 1\le i\le n-1\}.$$

Also, assume that $s_k=|X_k|$, the cardinal number of X_k. Our main goal in this paper is to find a way to calculate the value of s_k. The following theorem will give an inductive method for calculating the number s_k.

Theorem D. Assume that $\sigma \epsilon S_n$ is a permutation on n elements, for example $\{1,2,3, \dots, n\}$. Consider that k is an integer such that $1 \leq k < n$. We define

$$X_k = \{\sigma \epsilon S_n \mid \sigma(i+1) + k \neq \sigma(i) \; for \; 1 \leq i \leq n-1\}.$$

Also, assume that $s_k = |X_k|$, the cardinal number of X_k.

Then,

$$s_k = \binom{k}{1} s_{k-1} + \cdots + \binom{k}{k-1} s_1 + \binom{k}{k} s_0.$$

In this article, we try to deal with this problem with a probabilistic attitude. In fact, our goal is to be able to calculate a certain percentage of permutations that keep members constant for certain values. We will see that the solution of this problem is directly related to the derangement problem.

For more result, see [1], [2], [3], [4] and [5].

2 MAIN RESULT

Lemma 2.1. Assume that $S_n(\mathrm{r}) \subseteq S_n$. Also consider that $S_n(\mathrm{r})$ is the set of permutations of size n with r fixed points. Then:

$$|S_n(\mathrm{r})| = \binom{n}{\mathrm{r}} D_{n-\mathrm{r}}$$

Where, D_n is the number equal to the number of members of a derangement set of size n.

Proof.

It is enough to select r objects among n objects. There should be no constant member in the remaining set. Therefore, we will have a derangement with $n - r$ elements. Thus, we obtain that

$$|S_n(\mathrm{r})| = \binom{n}{\mathrm{r}} D_{n-\mathrm{r}}$$

As we desired.

Main Theorem. Assume that p is a real number such that $0 \leq p < 1$. Also, consider that $1 \leq r < n$. If the probability that a permutation with n elements has at least r fixed members is less than the number p, what relation should be established between r and n.

Solution.

The number of the permutations with n elements such that they have at least r fixed members is equal to:

$$|S_n(\mathrm{r})| + \cdots + |S_n(\mathrm{n})|$$

Thus, we must have

$$|S_n(\mathrm{r})| + \cdots + |S_n(\mathrm{n})| \leq (\mathrm{n}!)\mathrm{p}$$

Therefore,

$$\binom{n}{\mathrm{r}} D_{n-\mathrm{r}} + \cdots + D_0 \leq (\mathrm{n}!)\mathrm{p}$$

Consequently, we obtain that

$$\binom{n}{0} D_n + \cdots + \binom{n}{\mathrm{r}-1} D_{n-(r-1)} > (\mathrm{n}!)(1-\mathrm{p})$$

Now, Consider n sufficiently large. We now that:

$$\lim_{n\to\infty}\frac{D_n}{n!}=e^{-1}$$

So,

$$\lim_{n\to\infty}\frac{D_{n-r}}{n-r!}=e^{-1}$$

Thus, we have:

$$\frac{D_{n-r}}{n-r!}\approx e^{-1}$$

Hence,

$$|S_n(\mathrm{r})\ |=\binom{n}{\mathrm{r}}D_{n-\mathrm{r}}\approx\frac{\mathrm{n!}}{r!}e^{-1}$$

On the other hand,

$$\binom{n}{0}D_n+\cdots+\binom{n}{\mathrm{r}-1}D_{n-(r-1)}>(\mathrm{n!})(1-\mathrm{p})$$

Therefore,

$$\frac{\mathrm{n!}}{0!}e^{-1}+\cdots+\frac{\mathrm{n!}}{r-1!}e^{-1}>(\mathrm{n!})(1-\mathrm{p})$$

We obtain that

$$(\frac{1}{0!}+\cdots+\frac{1}{r-1!})e^{-1}>(1-\mathrm{p})$$

But, we know that

$$\lim_{r\to\infty}(\frac{1}{0!}+\cdots+\frac{1}{r-1!})e^{-1}=1$$

Then, these calculations will lead to the problem that we want to calculate the n umber from the above relation. Therefore, the number r is independent of the choice of the number n.

$$(\frac{1}{0!}+\cdots+\frac{1}{r-1!})>(1-\mathrm{p})\mathrm{e}$$

In special case, when $p=0.01$, we obtain that $r=5$, independent of the choice of the number n.

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REFERENCES

[1] M. Bóna, Combinatorics of Permutations, Chapman & Hall/CRC, 2004.
[2] Chenying Wang , Piotr Miska , István Mező, The r-derangement numbers, Discrete Mathematics, Vol. 340, 2017, 1681-1692.
[3] R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics, Addison Wesley, 1993.
[4] P. Miska, Arithmetic properties of the sequence of derangements, J. Number Theory 163 (2016) 114–145
[5]] M. Shattuck, Generalizations of Bell number formulas of Spivey and Mező, Filomat (in press)