Binomial Expansion and Fibonacci Series

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ABSTRACT

Assume that F_m, for $m \in \mathbb{Z}$, be the general sentence of the Fibonacci sequence.

Then:

$$\binom{m}{1}F_1 + \binom{m}{2}F_2 + \cdots + \binom{m}{m}F_m = F_{2m}.$$

And also,

$$\binom{m}{0}F_1 + \binom{m}{1}F_2 + \cdots + \binom{m-1}{m-1}F_m = F_{2m-1}.$$

KEYWORDS: Combinatorics, Fibonacci numbers, Partial permutation, Binomial expansion

1 INTRODUCTION

Neural networks use artificial neurons to understand and formulate human intelligence in the form of nested hierarchical concepts. At the same time humans and many things in nature obey Fibonacci sequence. Hence, in this article we aim to replicate a study back from 2000s claiming that Fibonacci initialized weight matrices with golden ratio as learning rate will outperform random initialization in terms of learning curve performance.

Fibonacci numbers or Fibonacci sequence is among the most popular numbers or sequence in mathematics. The sequence is in the form of 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... which first appeared in Liber Abaci book of Leonardo Pisano in 1202. It is often known as the Lame sequence or Viranka number as many other ancient mathematicians have used this sequence in their document. The first mathematician who called it Fibonacci sequence is Edouard Lucas in 19-th century. Lucas also showed that the Fibonacci sequence appears in the shallow diagonal of the Pascal triangle and he also defines a sequence based on the Fibonacci numbers, which is currently known as Lucas number. The complete information of the sequence can be found in the On-Line encyclopaedia of Integer Sequence.

One of the important features arising from the Fibonacci sequence is the Golden Ratio. It is the ratio of the consecutive numbers in the Fibonacci sequence which converges to 1.61803398875. The ratio has been found in many areas of applications such as in analyzing the proportions of natural objects and man-made systems. The ratio can also be found in modern applications such as financial analysis and plastic surgery. Given that it has many applications, many studies have been conducted to extend the sequence. The extension of the Fibonacci sequence is also widespread and penetrated many branches of mathematics including dynamical system. For example, the Fibonacci sequence has been extended to tribonacci, tetranacci, and other higher order n-nacci sequences. The n-nacci sequence has found application in coin tossing problem. On the other hand, the Fibonacci sequence also has been extended by generalizing the integer to real and complex numbers, quaternion, generalized quaternion.

In mathematics, the Fibonacci numbers, denoted by $\mathrm{F_n}$, form a sequence, called the Fibonacci sequence, such that each number is the sum of the two preceding ones, starting from 0 and 1. That is,

$$\mathrm{F}_{0,} = 0,\ \mathrm{F}_{1,} = 1,\ \mathrm{F_n} = \mathrm{F_{n-1}} + \mathrm{F_{n-2}} \text{ for n } > 1.$$

$$\{0,1,1,2,3,5,8,13,21,34,55,89,144,\dots\}.$$

$$\frac{1+\sqrt{5}}{2} = \varphi \simeq 1.618\dots \quad ,\frac{1-\sqrt{5}}{2} \simeq -0.618\dots$$

Johannes Kepler proved that $\lim\limits_{\mathrm{n}\to\infty} \frac{\mathrm{F_{n+1}}}{\mathrm{F_n}} = \varphi$.

Numerous other identities can be derived using various methods. Here are some of them:

Cassini's identity states that:

$$(\mathrm{F_n})^2 - \mathrm{F_{n+1}}\mathrm{F_{n-1}} = (-1)^{\mathrm{n}-1}.$$

Catalan's identity is a generalization:

$$(\mathrm{F_n})^2 - \mathrm{F_{n+r}}\mathrm{F_{n-r}} = (-1)^{\mathrm{n}-1}(\mathrm{F_r})^2.$$

d'Ocagne's identity

$$F_{n+1}F_m - F_{m+1}F_n = (-1)^n F_{m-\mathrm{n}}.$$

The last is an identity for doubling n; other identities of this type are:

$$5(F_n)^3 + 3(-1)^n F_n = F_{3n}.$$

In this field, you can also use combinational tools. The following theorems are examples of this matter:

Theorem A. Assume that $\mathrm{F_m}$, for m ∈ ℤ, be the general sentence of the Fibonacci sequence.

Then:

$$F_{2m+2} = \sum_{k=0}^{m}\sum_{s=0}^{m}\binom{m-k}{s}\binom{m-s}{k}$$

Theorem B. Assume that $\mathrm{F_n}$, for n ∈ ℤ, be the general sentence of the Fibonacci sequence.

Then:

$F_n = (-1)^n F_{-n} = \frac{1}{2}\left[-\binom{n}{1}F_0 + \binom{n}{2}F_1 - \cdots + (-1)^n \binom{n}{n-1} F_{n-1}\right]$, when n is an even number.

And also,

$\left[-\binom{n}{1}F_0 + \binom{n}{2}F_1 - \cdots + (-1)^n \binom{n}{n-1} F_{n-1}\right] = 0$, when n is an odd number.

In this article, we try to find a recursive relation based on binomial expansion for Fibonacci numbers using combinations tools and relations.

For more result, see [1], [2], [3], [4], [5] and [6].

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2 MAIN RESULT

Main Theorem. Assume that F_m, for $m \in \mathbb{Z}$, be the general sentence of the Fibonacci sequence.

Then:

$$\binom{m}{1}F_1 + \binom{m}{2}F_2 + \cdots + \binom{m}{m}F_m = F_{2m}.$$

And also,

$$\binom{m}{0}F_1 + \binom{m}{1}F_2 + \cdots + \binom{m-1}{m-1}F_m = F_{2m-1}.$$

Proof.

Let's use induction to prove this.

$$\binom{1}{1}F_1 = F_1 = 1 = F_2.$$

In addition,

$$\binom{1}{0}F_1 + \binom{1}{1}F_2 = 1 + 1 = 2 = F_3.$$

Now we assume that the proof is true for m based on induction assumption, we prove the correctness of the relation for $m+1$.

We know that

$$\binom{m-1}{r-1}F_1 + \binom{m-1}{r} = \binom{m}{r}.$$

Now, we have

$$A = \binom{m+1}{1}F_1 + \binom{m+1}{2}F_2 + \cdots + \binom{m+1}{m}F_m + \binom{m+1}{m+1}F_{m+1} =$$

$$A = \binom{m}{1} F_1 + \binom{m}{2} F_2 + \cdots + \binom{m}{m} F_m +$$

$$\binom{m}{0} F_1 + \binom{m}{1} F_2 + \cdots + \binom{m}{m-1} F_m + \binom{m}{m} F_{m+1}$$

Now according to the assumption of induction, we obtain that

$$A = F_{2m} + F_{2m+1} = F_{2m+2} = F_{2(m+1)}.$$

As we desired. Now notice that $F_0 = 0$. Let

$$B = \binom{m+1}{0} F_1 + \binom{m+1}{1} F_2 + \cdots + \binom{m+1}{m} F_{m+1} + \binom{m+1}{m+1} F_{m+2} =$$

$$B = \binom{m}{0} F_1 + \binom{m}{1} F_2 + \cdots + \binom{m}{m} F_{m+1} +$$

$$\binom{m}{0} F_2 + \binom{m}{1} F_3 + \cdots + \binom{m+1}{m} F_{m+1} + \binom{m}{m} F_{m+2}$$

Therefore,

$$B = \binom{m}{0} F_1 + \binom{m}{1} F_2 + \cdots + \binom{m}{m} F_{m+1} +$$

$$\binom{m}{0} (F_0 + F_1) + \binom{m}{1} (F_1 + F_2) + \cdots + \binom{m}{m} (F_m + F_{m+1})$$

Thus, we conclude that

$$B = \binom{m}{0} F_1 + \binom{m}{1} F_2 + \cdots + \binom{m}{m} F_{m+1} +$$

$$\binom{m}{0} F_0 + \binom{m}{1} F_1 + \cdots + \binom{m}{m} F_m +$$

$$\binom{m}{0} F_1 + \binom{m}{1} F_2 + \cdots + \binom{m}{m} F_{m+1}$$

Consequently,

$$B = 2(\binom{m}{0} F_1 + \binom{m}{1} F_2 + \cdots + \binom{m}{m} F_{m+1}) +$$

$$\binom{m}{0} F_0 + \binom{m}{1} F_1 + \cdots + \binom{m}{m} F_m +$$

Now according to the assumption of induction, we obtain that

$$B = 2F_{2m+1} + F_{2m}$$

$$B = F_{2m+1} + F_{2m+1} + F_{2m}$$

$$B = F_{2m+1} + F_{2m+2}$$

$$B = F_{2m+3}.$$

As we claimed.

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