Application of Young Diagram in the Partition Problem

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ABSTRACT

Assume that m is a natural number. Also, consider that s, t are two natural numbers. Then, the number of partitions of the number m in which the number s appears at least t times is equal to the number of partitions of the number m in which the number t appears at least s times.

KEYWORDS: Combinatorial method, Permutation, Partition, Young diagram.

1 INTRODUCTION

In number theory and combinatorics, a partition of a non-negative integer n, also called an integer partition, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. (If order matters, the sum becomes a composition.) For example, 4 can be partitioned in five distinct ways.

Partitions can be graphically visualized with Young diagrams or Ferrers diagrams. They occur in a number of branches of mathematics and physics, including the study of symmetric polynomials and of the symmetric group and in group representation theory in general. There are two common diagrammatic methods to represent partitions: as Ferrers diagrams, named after Norman Macleod Ferrers, and as Young diagrams, named after the British mathematician Alfred Young. Both have several possible conventions; here, we use English notation, with diagrams aligned in the upper-left corner.

The asymptotic growth rate for $p(n)$ is given by:

$$\log\big(p(n)\big) \sim C\sqrt{n} \ \ as\ n \to \infty$$

Where,

$$C = \pi\sqrt{\frac{2}{3}}.$$

The more precise asymptotic formula (see [1]):

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) as\ n \to \infty,$$

In this regard, the following theorems have been proved.

Theorem A. Assume that A is a set with M elements. Also, consider that we intend to divide this set into p partitions. Some subsets may even be empty. Therefore, the value of mathematical expectation of members of the largest subset is equal to:

$$\left(\sum_{i=1}^{M}\sum_{j=1}^{p}(-1)^{j+1}\binom{p}{j}\frac{\binom{M+p-1-ij}{p-1}}{\binom{M+p-1}{p-1}}\right)-1.$$

Theorem B. Assume that A is a set with m elements.

$$\mathrm{A}=\{1,2,\dots,\mathrm{m}\}$$

Also, assume that consider that $\mathrm{A_t}$ be the set of all partitions of A whose number of members in each section is less than or equal to t.
In this case, we intend to calculate the number of members of this set. Also in the special case when $\mathrm{t}=2$, we show that

$$\mathrm{cardinal}(\mathrm{A_2})=\mathrm{cardinal}(\mathrm{B_2})+(m-1)\mathrm{cardinal}(\mathrm{C_2})$$

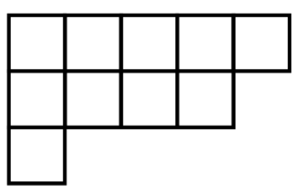
When:

$$\mathrm{B}=\{1,2,\dots,\mathrm{m}-1\}$$
$$\mathrm{C}=\{1,2,\dots,\mathrm{m}-2\}$$

Also,

$$|\mathrm{A_2}\ |=\sum_{k=o}^{[\frac{m}{2}]}\frac{\mathrm{m}!}{k!\,(m-2k)!\,2^k}$$

An alternative visual representation of an integer partition is its Young diagram (often also called a Ferrers diagram). Rather than representing a partition with dots, as in the Ferrers diagram, the Young diagram uses boxes or squares. Thus, the Young diagram for the partition $5+4+1$ is:



While this seemingly trivial variation does not appear worthy of separate mention, Young diagrams turn out to be extremely useful in the study of symmetric functions and group representation theory: filling the boxes of Young diagrams with numbers (or sometimes more complicated objects) obeying various rules leads to a family of objects called Young tableaux, and these tableaux have combinatorial and representation-theoretic significance. As a type of shape made by adjacent squares joined together, Young diagrams are a special kind of polyomino.

There is a natural partial order on partitions given by inclusion of Young diagrams. This partially ordered set is known as Young's lattice. The lattice was originally defined in the context of representation theory, where it is used to describe the irreducible representations of symmetric groups Sn for all n, together with their branching properties, in characteristic zero. It also has received significant study for its purely combinatorial properties; notably, it is the motivating example of a differential poset.

For more, see [1],…,[7].

2 MAIN RESULT

Theorem 2-1. Assume that m is a natural number. Also, consider that s, t are two natural numbers. Then, the number of partitions of the number m in which the number s appears at least t times is equal to the number of partitions of the number m in which the number t appears at least s times.

Proof.

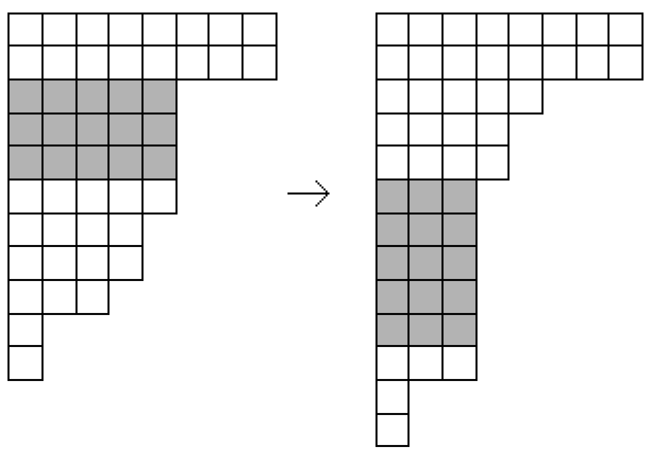
First, assume that $m \leq st$, in this case the proof is clear. Now, assume that $st < m$. Take the set A as the set of all partitions of m such that in which the number s appears at least t times. Now, Take the set B as the set of all partitions of m such that in which the number t appears at least s times.

We must show that there is a one-to-one correspondence between these two sets. If in a partition the number s appears at least t times, then we have a rectangle with st elements. Also, If in a partition the number t appears at least s times, then we have a rectangle with st elements.

Now leave this rectangle in two cases. In the remaining members, a one-to-one correspondence between the number of secretions can be established.

In fact, this problem leads to the fact that the number of partitions of the number $m - st$ in which the number s appears at least 0 times is equal to the number of partitions of the number $m - st$ in which the number t appears at least 0 times.

The diagram below can give us a good visualization of the removal of the mentioned rectangle. Practically, a one-to-one correspondence can be established to the remaining houses in two cases.



This diagram helped shorten our proof. If we want to prove this problem algebraically or for example by mathematical induction, the proof will not be so short and simple. This issue shows us the importance of using Young's diagram.

Proceed with the same proof by induction. When you add a member to the previous set, new situations may occur that will complicate the problem solving.

Somehow, in this article, we tried to show the importance of this type of proof over algebraic and computational proofs.

3 ACKNOWLEDGEMENTS

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