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Characterizing spanning trees which give a minimum fundamental cycle basis for a special family of graphs

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Abstract

Let $G = (V, E)$ be a simple graph Any cycle in a simple graph with vertex set V and edge set E. Then any cycle C in G can be considered as an incidence vector of size $e = |E|$. The set of all linear combination of these vectors is called the cycle space of G. A basis for cycle space is called cycle basis of G. A cycle basis B is called fundamental if there exists a spanning tree T of G such that any member C of B is a cycle which has exactly one edge from $E \setminus T$. In this paper for special family of graphs we characterize all trees which build a minimum fundamental cycle basis.

Keywords: Cycles in graphs, Cycle basis, Minimum cycle basis.

AMS Mathematical Subject Classification [2010]: 05C50, 05C90

1 Introduction

Cycle in graph is an important structure of graph which appear in many areas of mathematics, science and engineering. For instance cycles paly important rule in periodic scheduling, graph drawing, analysis of chemical and biological pathways and , analysis of electrical networks. Cycle space of graph is a linear space which contains all cycles of graphs and also all linear combination of cycles, which is considered over different fields. A cycle basis is a basis of cycle space. In fact a cycle basis is a compact representation of the set of all cycles in graph. There are different types of cycle basis. More precisely strictly fundamental, weakly fundamental, totally unimodular, integral, directed and undirected basis are some kinds of cycle basis of graphs. In this paper we we consider some kinds of cycle basis for special family of graphs.

A cycle of a graph $G = (V, E)$ is a connected regular subgraph of degree 2. Any cycle in G can be represented by an incidence vector $\gamma_C \in \{0, 1\}^{|E|}$ ($\gamma_C \in \{0, \pm 1\}^{|E|}$ in directed case). The *cycle space* of G is the vector space generated by $\{\gamma_C |\ C \text{ is a cycle in } G\}$ over $\mathbb{Z}_2$ (over $\mathbb{Q}$ in directed case). A *cycle basis* for G consists of some cycles which form a basis for cycle space of G. The *length of a cycle basis* is the total length of the cycles included in the basis. A *minimum cycle basis* (or MCB for short) of a graph is a cycle basis with minimum length. In [5] the authors give a good survey on cycle basis of graphs. In [7] five different classes of cycle bases are defined.

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Definition 1.1. Let $B = \{C_1, C_2, \ldots, C_\nu\}$ be a directed cycle basis for a graph G, then B is called fundamental if there exits some spanning tree T of G such that $\mathcal{B} = \{C_T(e) | e \in E(G) \setminus E(T)\}$, where $C_T(e)$ denoted the unique cycle in $T \cup \{e\}$.

It is easy to see that every fundamental basis is weakly fundamental and every weakly fundamental is integral basis. But finding minimum basis in each class is an interesting problem. The problem of finding minimum weakly fundamental basis is an APX-hard problem. Hence, solving this problem for special family of graphs is interesting. In this paper we find a minimum cycle basis for some special graph products.

2 Minimum Fundamental Basis

Let $K_{n,n}$ be a complete bipartite graph of order $2n$ and M be a complete matching of $K_{n,n}$. Suppose G_n be a graph constructed from $K_{n,n} \setminus M$. Then for each n we will find a minimum weakly fundamental basis for graph G_n. The proof is mostly based on work in [3].

In this section we characterize all trees which give an MFCB for G_n. For this aim we need the following theorem of [3], in which the authors compute the length of an MFCB of G_n.

Theorem 2.1. [3] *For any integer $n \geq 3$, a minimum fundamental cycle basis for graph G_n consists of one cycle of length 6 and all other cycles of length 4.*

Let T be such a tree and $\mathcal{B}$ be the corresponding MFCB. Let f be a function on $\{1, 2, \ldots, 2n\}$ defined as:

$$f(x) = \begin{cases} x + n & \text{if} \quad x \in \{1, 2, \ldots, n\}, \\ x - n & \text{if} \quad x \in \{n + 11, n + 2, \ldots, 2n\}. \end{cases}$$

In the following lemmas we prove that the diameter of T should be bounded.

Lemma 2.2. *Tree T does not contain any path of length 7, as a subtree.*

Proof. For the contrary suppose that T contains path P of length 7. Without loss of generality we can suppose that $P = a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4$, where $a_i \in A$ and $b_i \in B$, for any $i \in \{1, 2, 3, 4\}$. By Theorem 2.1, $\mathcal{B}$ does not have a cycle of length 8, so $b_4 a_1 \notin E(G_n)$ and since $f(a_1)$ is the only vertex in A which does not connect to a_1 in G_n we have $b_4 = f(a_1)$. Hence, $b_3 \neq f(a_1)$, which means that $a_1 b_3 \in E(G_n)$ and $C_1 = (a_1, b_1, a_2, b_2, a_3, b_3)$ is the unique longest cycle of G_n and all its other cycles have length 4. Since $a_2 \neq a_1 = f(b_4)$, we have $a_2 b_4$ is an edge of G_n, adding which to T gives a cycle of length 6 in $\mathcal{B}$ other than C_1. It is a contradiction.

□

Lemma 2.3. *Tree T does not contain any path of length 6.*

Proof. For the contrary suppose that T contains a path of length 7 called P. Without loss of generality we can suppose that $P = a_1, b_1, a_2, b_2, a_3, b_3, a_4$, where, $a_i \in A$ and $b_j \in B$, for any $i \in \{1, 2, 3, 4\}$ and $i \in \{1, 2, 3\}$. Using Theorem 2.1, we conclude that graph G_n has atmost one edge form the set $\{a_1 b_3, a_4 b_1\}$ so $a_4 = f(b_1)$ or $a_1 = f(b_3)$. Without loss of generality suppose that $a_4 = f(b_1)$. Suppose that $B_1, B_2, B_3 \in \mathcal{B} \setminus \{b_1, b_2, b_3, f(a_1)\}$. Now, before completing the proof of this lemma we need to prove following proposition:

Proposition 2.4. *With the same notation as in the proof of Lemma 2.3 the followings hold:*

(i) *Non of the vertices B_1, B_2 and B_3 can be connected to b_1 or b_3 or b_2 in T, by an edge disjoint path from P.*

(ii) *At most one of the vertices B_1, B_2 or B_3 can be connected to a_2 in T, by an edge disjoint path from P.*

(iii) *At most one of the vertices B_1, B_2 or B_3 can be connected to a_3 in T, by an edge disjoint path from P.*

Proof.

(i) If one of the B_1, B_2 and B_3 connect to b_1 or b_3 by such a path, called Q, then the length of Q is at least 2 and hence T contains a path of length at least 7 which is contrary with Lemma 2.2. Now, suppose that at least on of the B_1, B_2 or B_3 connected to b_2 in T, with edge disjoint path Q from P. Without loss of generality suppose that B_1 connect to b_2 by q. Since $B_1 \neq f(a_1)$ and $B_1 \neq f(a_4)$, we have $\{B_1a_1, B_1a_4\} \subset E(G_n)$. Now, adding B_1a_1 and B_1a_4 respectively to T gives two different cycles of length 6 in $\mathcal{B}$, which contradicts with Theorem 2.1.

(ii) For the contrary suppose that two of the B_i's, called B_s and B_t, connect to a_2, by such paths, Now adding the edges a_4B_2 and a_4B_3 from G_n to T, gives two cycles of length at least 6 in $\mathcal{B}$, which is a contradiction with Theorem 2.1.

(iii) The proof is exactly the same as the proof of (ii), when replacing a_2 by a_3. □

Now, we complete the proof of Lemma 2.3. Using Proposition 2.4 (ii) and (iii), without loss of generality we can suppose that B_1 does not connect to neither a_2 nor a_3 by an edge disjoint path Q from P, in T. On the other hand by Proposition 2.4 (i), B_1 can not be connected to non of the b_1, b_2 and b_3. So, B_1 should be connected to a_1 or a_4 by an edge disjoint path Q from P, which give a path of length at least 7 in T, which contradicts with Lemma 2.2. Hence, T does not have a path of length 6 as a subtree. □

Theorem 2.5. *Let T be a tree which gives an MFCB for G_n. Then T is isomorphic to one of the trees T_1, T_2 or T_3 shown in Figure 1.*

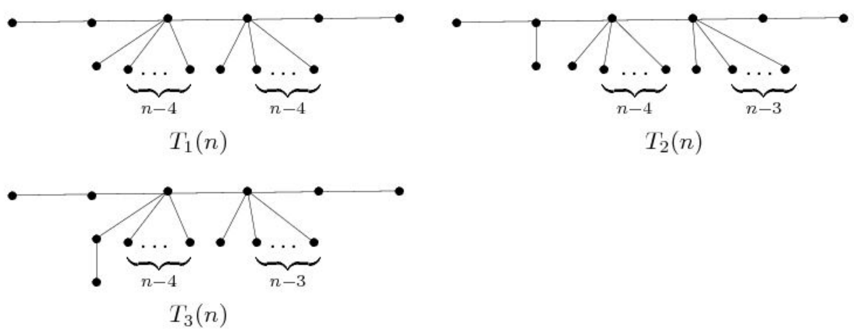

Figure 1: Trees which give MFCB for G_n

Proof. By Theorem 2.1 $\mathcal{B}$ has a unique cycle of length 6. So T contains a path P of length 6 with end points a_1 and b_3 and $a_1b_3 \in E(G_n)$. Suppose that $p = a_1, b_1, a_2, b_2, a_3, b_3$. Fist we note that, by Lemma 2.3, there is not any edge disjoint path from P of length more than 1 having b_1 or a_3 as an end point. Now, suppose that b_1 has a neighbor outside of P, then it should be $f(b_1)$ (otherwise $\mathcal{B}$ has a second cycle of length 6). Similarly, if a_3 has a neighbor outside of P, then it should be $f(a_3)$. Using Lemma 2.3, there is not any edge disjoint path from P of length more than 2 having b_2 or a_2 as an end point. And if, there is an edge disjoint path $Q_1 = a_2, b, a$ from P of length 2 then a should be $f(b_3)$ and with a similar discussion if, there is an edge disjoint path $Q_2 = b_2, a', b'$ from P of length 2 then b should be $f(a_1)$. It is easy to check that non two case of these four cases can occur at the same time, otherwise a second cycle of length 6 is in $\mathcal{B}$, which is a contradiction. If non of these four cases occur then $T = T_1$, if case 2, or case 3, occur, than $T = T_2$, finally, if case 3 or case 4 occur, then $T = T_3$. Moreover it is easy to check that trees T_1, T_2 and T_3 gives an MFCB for G_n.

□

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