8th International Conference on
Combinatorics, Cryptography, Computer Science and Computation
November. 15-16, 2023

On Tree Graphs and Arboreal Hypergraphs

Roohallah Daneshpayeh[1]
Department of Mathematics, University of Payame Noor, Tehran, Iran

Sirus Jahanpanah
Department of Mathematics, University of Payame Noor, Tehran, Iran

Abstract

This paper presents the notation of quasihypergraphs and computes the number of the set of all quasihypergraphs that are constructed on any given nonvoid set. The set of hypergraphs and arboreal hypergraphs are special subclasses of quasihypergraphs in which we compute the number of members of these classes via the recursive sequences. This study tries to extract a graph from arboreal hypergraph as derivable graphs via positive relations and the notion path in arboreal hypergraph. Also, it is considered the numerical relationship between the number of derivable graphs and the number of arboreal hypergraphs. The notation of valued-part in arboreal hypergraphs which is presented in this study, plays an important role in the computation of derivable graphs.

Keywords: Arboreal hypergraph, quasihypergraph, valued-section, positive relation.

AMS Mathematical Subject Classification [2010]: 05C65, 05C05

1 Introduction

In today's world, most of the real problems are based on modeling graphs and graph trees. The importance of graphs and graph trees in all engineering sciences, especially computer engineering, is not hidden from anyone. Designing complex networks based on graphs is one of the most important issues in decision-making today, which has many applications in all fields of natural sciences, mathematics, and engineering. Since graphs have limitations in the collective communication of objects and cannot connect more than two elements at the same time, the idea of hypergraphs can be particularly important in overcoming this limitation. The structure of hypergraphs has been presented by Berge as an extension of the theory of graphs with this the motivation that hypergraphs cover problems and shortcomings of structures of the graph around 1960 [2]. Hypertrees or tree hypergraphs are specially connected hypergraphs that are a generalization of trees. In the structure of hypergraphs, we can connect one group of elements with another group of elements, and in addition, the relationship of except to except, except to whole, and whole to whole objects is investigated. In this research, considering the importance of the theory of hypergraphs, we specifically address the concept of hypertrees, and considering the special importance of tree graphs, we try

[1]speaker

to create a precise connection between tree graphs and hypertrees. We have been able to extract a tree graph from each hypertree with special features and we have discussed its basic properties. Our main motivation for this idea is to first consider a complex hypernetwork of a real problem model as a hypertree and create an induction relation on this hypertree based on our goals. In fact, with this technique, we convert a complex network into a tree graph that fulfills all our goals in this problem, and after that, engineers specializing in programming can use the algorithms extracted from the mathematical results of that program code. To learn more about the importance of this topic, one can read the following research, spectral moments of hypertrees and their applications [3], topology and geometry of random 2−dimensional hypertrees [5], on the irregularity of uniform hypergraphs [6], steiner connectivity problems in hypergraphs [4], embedding wheel-like networks [7], computing optimal hypertree decompositions with SAT [8], distance-based topological descriptors on ternary hypertree networks [9] and a vulnerability measure of k-uniform linear hypergraphs [10].

Definition 1.1. [2] Let X be a finite set and $P^*(X) = \{Y \mid \emptyset \neq Y \subseteq X\}$. A *hypergraph* on X is a pair $H = (X, \{E_i\}_{i=1}^m)$ such that for all $1 \leq i \leq m, E_i \in P^*(X)$ and $\bigcup\limits_{i=1}^m E_i = X$. The elements $x_1, x_2, \ldots, x_n$ of X are called *hypervertices*, and the sets $E_1, E_2, \ldots, E_m$ are called the *hyperedges* of the hypergraph H.

Definition 1.2. [1] Let $H = (X, \{E_i\}_{i=1}^m)$ be a hypergraph. A walk of length l in a hypergraph H is a sequence $x_1\ E_1\ x_2\ E_2\ x_3\ E_3 \ldots\ x_l\ E_l\ x_{l+1}$ such that for all $i \in \{1, 2, \ldots, l\}, x_i, x_{i+1} \in E_i$. A walk of length l in a H is said to be a path if, (i) all the vertices $x_1, x_2, ..., x_{l+1}$ except x_1 and x_{l+1} are distinct and (ii), all the edges $E_1, E_2..., E_l$ are distinct(we denote the path between of x, y by $P(x, y)$). If $l > 1$ and $x_1 = x_{l+1}$, the path $x_l\ E_1\ x_2\ E_2\ x_3 \ldots\ x_l\ E_l\ x_{l+1}$ is called a cycle of length l. A hypergraph H is connected if for any two vertices $a, b \in X$ there is a path joining the vertices a and b. A hypergraph $H = (X, \{E_i\}_{i=1}^m)$ is called an arboreal hypergraph, if it is connected, non-trivial and cycle-free.

2 On quasihypergraphs

In this section, we present the notion of quasihypergraphs and investigate of their properties. Also we compute the number of the set of all quasihypergraphs which are constructed on any given nonvoid set.

Definition 2.1. Let X be a finite set. A *quasihypergraph* on X is a pair $H = (X, \{E_k\}_{k=1}^m)$, such that for all $1 \leq k \leq m$, $\emptyset \neq E_i \subseteq X$.

From now on, we will denote the set of all quasihypergraphs on X by $\mathcal{SH}(X)$ and the number of elements of any given set U by $Car(U)$.

Lemma 2.2. *Let X be a nonvoid set. Then* $Car(\mathcal{SH}(X)) = 2^{2^{Car(X)-1}} - 1$.

Theorem 2.3. *Let X be a nonvoid set. If* $\bigcup\limits_{i=1}^m E_i = X$, *then* $Car(\mathcal{SH}(X)) = \sum\limits_{k=0}^{Car(X)} (-1)^k \binom{Car(X)}{k} 2^{\epsilon_k}$, *which* $\epsilon_k = 2^{(Car(X)-k)} - 1$.

Skeleton Proof 2.4. *Let $Car(X) = k$ and $a_k = Car(\mathcal{SH}(X))$. Now, we compute the a_k by the recursive sequences. Let $X = \{x_1, x_2, \ldots, x_k\}$ and $S_i = \{Y \in \mathcal{SH}(X_k) \mid X_k = X \setminus \{x_1, x_2, \ldots, x_i\}\}$. One can* *see that* $Car(\mathcal{SH}(X)) = Car((\bigcup_{i=1}^{k} S_i)^c), Car(S_i) = a_{k-i}$ *and so* $a_k = \sum_{k=0}^{Car(X)} (-1)^k \binom{Car(X)}{k} 2^{\epsilon_k}$, *which* $\epsilon_k = 2^{(Car(X)-k)} - 1$.

Corollary 2.5. *Let X be a nonvoid set. Then for* $\epsilon_{k,i} = 2^{(Car(X)-i-k)} - 1$,

$$1 + \sum_{i=0}^{Car(X)-1} \binom{Car(X)}{i} \sum_{k=0}^{Car(X)-i} (-1)^k \binom{Car(X)-i}{k} 2^{\epsilon_{k,i}} + \sum_{k=0}^{Car(X)} (-1)^k \binom{Car(X)}{k} 2^{\epsilon_k} = 2^{2^{Car(X)-1}}.$$

2.1 On k-separated quasihypergraphs and arboreal hypergraphs

In this subsection, we introduce the notation of k-separated quasihypergraphs and compute the number of the set of all k-separated quasihypergraphs which are constructed on any given nonvoid set. Also we introduce a positive relation on arboreal hypergraphs which converts any arboreal hypergraph to a graph or arboreal graph.

Definition 2.6. Let X be a nonvoid set and $k \in \mathbb{N}$. Then $H = (X, \{E_i\}_{i=1}^{m})$ is called a k-separated quasihypergraph, if for every $1 \leq i \neq j \leq m, E_i \cap E_j = \emptyset$ and $\bigcup_{i=1}^{m} E_i = X$. We will denote the set of k-separated quasihypergraph on X, by $\mathcal{SH}^{(k)}(X)$.

Corollary 2.7. *Let X be a nonvoid set and $k \in \mathbb{N}$. Then* $Car(\mathcal{SH}^{(k)}(X)) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^{Car(X)}$.

Theorem 2.8. *Let $H = (X, \{E_i\}_{i=1}^{m})$ be an arboreal hypergraph. Then for all $1 \leq t \leq m$, there exists $1 \leq s \leq m$, such that $E_t \cap E_s \neq \emptyset$.*

Proof. Let $1 \leq t \leq m$. Since $H = (X, \{E_i\}_{i=1}^{m})$ is an arboreal hypergraph, there exists $x \in E_t$ and so for all $y \in X$, there exists $E_{t'}$ such that $y \in E_{t'}$. Now, $H = (X, \{E_i\}_{i=1}^{m})$ is a connected hypergraph, thus there is a sequence $x = x_1\ E_1\ x_2\ E_2\ x_3\ E_3 \ldots\ x_l\ E_l\ x_{l+1} = y$ such that for all $i \in \{1, 2, \ldots, l\}, x_i, x_{i+1} \in E_i$. It follows that for all $1 \leq t \leq m$, there exists $1 \leq s \leq m$, such that $E_t \cap E_s \neq \emptyset$. □

Definition 2.9. Let $H = (X, \{E_i\}_{i=1}^{m})$ be an arboreal hypergraph and $x \in X$. Define $N(x, E_i) = \{y \in X \mid \{x, y\} \subseteq E_i\}$ and a relation R on H by $(x, y) \in R$ if and only if $N(x, E_i) = N(y, E_i)$ and for any $i \neq j, \emptyset = N(y, E_j)$.

Theorem 2.10. *Let $H = (X, \{E_i\}_{i=1}^{m})$ be an arboreal hypergraph. Then there exists a binary operation $\Theta^{(\lambda)}$ on $R(H)$ such that $(R(H), \Theta^{(\lambda)})$ is a graph.*

Skeleton Proof 2.11. *Let $R(x), R(y) \in R(H)$ and $\lambda \in \mathbb{N}$. If there exists a path $P(x, y)$ such that $Car(P(x, y)) = \lambda$, then define $\Theta^{(\lambda)}(R(x), R(y)) = e_{R(x),R(y)}$, which $e_{R(x),R(y)}$ shows that there exists an edge between of $R(x), R(y)$ and $\Theta^{(\lambda)}(R(x), R(y)) = e_{\emptyset}$ shows that dose not exist any edge between of $R(x), R(y)$.*

From now on, for any given set A, B, if $A \cap B \neq \emptyset$ and $Car(A \cap B) = n$, we denote it by $A \overset{n}{\approx} B$.

Theorem 2.12. *Let $H = (X, \{E_i\}_{i=1}^{m})$ be an arboreal hypergraph and $x, y \in X$. If $E_i \overset{1}{\approx} E_j$, then* $\dfrac{Car(P(R(x), R(y)))}{\lfloor \frac{Car(P(x,y))}{\lambda} \rfloor} = 1$.

Theorem 2.13. *Let $H = (X, \{E_i\}_{i=1}^m)$ be an arboreal hypergraph. If for any $E_i \in E$ there exists $E_j \neq E_i$ such that $E_i \overset{\geq 2}{\approx} E_j$, then $(R(H), \Theta^{(\lceil \frac{m}{2} \rceil)})$ is a connected graph.*

Let $H = (X, \{E_i\}_{i=1}^m)$ be an arboreal hypergraph and $k \geq 3$. We will denote $\mathcal{P}^k(H) = \{E_i \in P^*(X) \mid E_i \text{ is a k}-part\}$.

Theorem 2.14. *Let $H = (X, \{E_i\}_{i=1}^m)$ be an arboreal hypergraph and $k \geq 3$. If $Car(\mathcal{P}^k(H)) = \beta$, then $Car(E(R(H), \Theta^{(1)})) \geq (\beta(\binom{k}{2} - 1) + m)$.*

Skeleton Proof 2.15. *Let E_s is a k-part hyperedge in an arboreal hypergraph $H = (X, \{E_i\}_{i=1}^m)$. Then there exist at least $k-1$ hyperedges $E_{i_1}, E_{i_2}, \ldots, E_{i_{k-1}}$, and $x_s \in E_s$ such that $E_s = \{x_s\} \cup \bigcup_{j=1}^{k-1} (E_s \cap E_{i_j})$, for all $1 \leq j \leq k-1, x_s \notin E_{i_j}$ and for all $1 \leq j' \neq j \leq k-1, E_s \cap (E_{i_j} \cap E_{i_{j'}}) = \emptyset$.*

Theorem 2.16. *Let $H = (X, \{E_i\}_{i=1}^m)$ be a complete hypergraph. If for all $1 \leq i \neq j \ \leq m, E_i \overset{k}{\approx} E_j$, then $(R(H), \Theta^{(1)}) \cong K_{\lfloor \frac{Car(X)}{k} \rfloor}$;*

Theorem 2.17. *Let $H = (X, \{E_i\}_{i=1}^m)$ be an arboreal hypergraph and $1 \leq i \neq j \ \leq m$. Then*
$$E_i \overset{Car(E_i \setminus E_j) = Car(E_j \setminus E_j) = k}{\approx} E_j \text{ and } E_i \not\subseteq E_j, \text{ imply } \frac{Car(E(R(H)))}{\lceil \frac{1}{2k} \sum_{i=1}^{m} Car(E_i) \rceil} = 1.$$

Skeleton Proof 2.18. *Let $x \in X$. Then there exist E_i, E_j such that either $x \in (E_i \setminus E_j) \setminus (E_j \setminus E_i)$, $1 \leq i \neq j \ \leq m, Car(E_i \setminus E_j) = Car(E_j \setminus E_j) = k$ or $E_i \overset{k}{\approx} E_j$. Thus we get that $Car(R(x)) = \dfrac{Car(E_i)}{2k}$.*

3 Discussion of results and conclusion

The current paper has introduced the limitations of graphs and trees and introduce the notion of quasi-hypergraphs. The main motivation of this study is make a connection between of trees and hypertrees, so we present a positive relation on hypertrees. This study is important in application of hypergraphs on the problems of hypernetworks in all sciences, special computer sciences.

Acknowledgment

The authors wish to thank the reviewers for their excellent suggestions that have been incorporated into this paper.

References

[1] I. Adler„ G. Gottlob, M. Grohe, *Hypertree width and related hypergraph invariants*, European Journal of Combinatorics, 28(8) (2007) 2167–2181.

[2] C. Berge, *Graphs and Hypergraphs*, North Holland, 1979.

[3] L. Chen, C. Bu, & J. Zhou, *Spectral moments of hypertrees and their applications*, Linear and Multilinear Algebra, 70(21) (2022) 6297–6311.

[4] F. Horsch, Z. Szigeti, *Steiner connectivity problems in hypergraphs*, Information Processing Letters, 183 (2024) 106428.

[5] M. Kahle and A. Newman, *Topology and Geometry of Random 2−Dimensional Hypertrees*, *Discrete. Comput. Geom.*, 67 (2022) 1229–1244.

[6] H. Lin, G. Yu, B. Zhou, *On the irregularity of uniform hypergraphs*, Linear Algebra and its Applications, 678(1) (2023) 107–124.

[7] R. S. Rajan, T. M. Rajalaxmi, S. Stephen, A. A. Shantrinal, K. J. Kumar, *Embedding Wheel-like Networks*, 18(2) (2023) 185–198.

[8] A. Schidler, S. Szeider, *Computing optimal hypertree decompositions with SAT*, Artificial Intelligence, 325 (2023) 104015.

[9] Y. Yu, D. A. Xavier, E. S. Varghese, D. Mathew, M. K. Siddiqui, and S. A. Fufa, *Distance-Based Topological Descriptors on Ternary Hypertree Networks*, Complexity, (2022), 9 pages.

[10] N. Zhao, H. Zhao, Y. Li, *A Vulnerability Measure of k-Uniform Linear Hypergraphs*, Symmetry 15(6) (2023) 1187.

e-mail: `rdaneshpayeh@pnu.ac.ir`
e-mail: `s.jahanpanah@pnu.ac.ir`