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A note on total limited packing in graphs

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Abstract

Let $G = \big(V(G), E(G)\big)$ be a graph. A set $B \subseteq V(G)$ is said to be a *k-total limited packing* in the graph G if $|B \cap N(v)| \leq k$ for each vertex v of G. The *k-total limited packing number* $L_{k,t}(G)$ is the maximum cardinality of a k-total limited packing in G.

Here we prove some results on the k-total limited packing numbers for graphs with emphasis on trees, specially when $k = 2$. Also we give some lower and upper bounds for this parameter.

Keywords: open packing, k-total limited packing number, total domination

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1 Introduction

Throughout this manuscript, we consider G as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of graph is denoted by n and the *size* of graph is m.

The *open neighborhood* of a vertex v is denoted by $N(v)$, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The subgraph induced by $S \subset V(G)$ in a graph G is denoted by $G[S]$.

A set $S \subseteq V(G)$ is a *dominating set* in the graph G if every vertex not in S has a neighbor in S. The *domination number*, denoted $\gamma(G)$, is the smallest number of vertices in a dominating set. A set $S \subseteq V(G)$ is a *total dominating set* in the graph G if every vertex in $V(G)$ is adjacent to an element of S. The *total domination number*, denoted $\gamma_t(G)$, is the smallest number of vertices in a total dominating set.

A set of vertices $B \subseteq V(G)$ is called a *packing* (resp. an *open packing*) in G provided that $N[u] \cap N[v] = \emptyset$ (resp. $N(u) \cap N(v) = \emptyset$) for each distinct vertices $u, v \in V(G)$. The maximum cardinality of a packing (resp. open packing) is called the *packing number* (resp. *open packing number*), denoted $\rho(G)$ (resp. $\rho_o(G)$). For more information about these topics, the reader can consult [5] and [6]. In 2010, Gallant et al. ([4]) introduced the concept of limited packing in graphs. In fact, a set $B \subseteq V(G)$ is said to be a *k-limited packing* (kLP) in the graph G if $|B \cap N[v]| \leq k$ for each vertex v of G. The *k-limited packing number* $L_k(G)$ is the maximum cardinality of a kLP in G. They also exhibited some real-world applications of it in network security, market situation, NIMBY and codes. This concept was next investigated in many papers, for instance, [2, 3, 10]. Similarly, a set $B \subseteq V(G)$ is said to be a *k-total limited packing* (kTLP)

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if $|B \cap N(v)| \leq k$ for each vertex v of G. The *k-total limited packing number* $L_{k,t}(G)$ is the maximum cardinality of a kTLP in G. This concept was first studied in [7] and some theoretical applications of it were given in [8, 1]. It is easy to see that the latter two concepts are the same with the concepts of packing and open packing when $k = 1$.

Here we give some lower and upper bounds for kTLP . Several sharp inequalities concerning this parameter are given with emphasis on trees, specially when $k = 2$.

For the sake of convenience, for any graph G by an $\eta(G)$-set with $\eta \in \{L_k, \gamma_t, \rho, \rho_o, L_{k,t}\}$ we mean a kLP set, TD set, packing set, open packing set and kTLP set in G of cardinality $\eta(G)$, respectively.

2 Main results

Let G be a graph of order n. If $k \geq n-1$, then $L_{k,t}(G) = n$. Note that the above condition that $k \geq n-1$ can be weakened to $k \geq \Delta(G)$. So, we only need to consider the k-TLP number for graphs G for which $k < \Delta(G)$.

Let G be a graph of order at least n. Then, $k \leq L_{k,t}(G) \leq n$. In the next theorem, we give an upper bound for the k-total limited packing number of a graph.

Theorem 2.1. *Let G be a graph of order n. Then, $L_{k,t}(G) \leq n + k - \Delta(G)$.*

Proof. Let v' be a vertex of maximum degree in G. If $k \geq \Delta(G)$, then it is obvious that $V(G)$ is a k-TLP set of G. Thus, $L_{k,t}(G) = n \leq n + k - \Delta(G)$. Hence, we assume that $k < \Delta(G)$. Let S be an $L_{k,t}(G)$-set. Since $|N(v') \cap S| \leq k$, there are at least $\Delta(G) - k$ vertices in $N(v')\backslash S$. So, $|\overline{S}| \geq \Delta(G) - k$. Therefore, $L_{k,t}(G) = |S| = n - |\overline{S}| \leq n + k - \Delta(G)$. □

We define the ζ family consisting of all graphs G constructed as follows.
Let G be a graph of order n so that $V(G) = A \cup B$ has the following conditions:

(i) $|A \cap B| = 3$,

(ii) $G[A]$ has a spanning star, and each component of $G[B]$ is a path,

(iii) $|N(v) \cap B| \leq 2$ for every vertex $v \in \overline{B}$.

Figure 1 depicts a representative member of ζ.

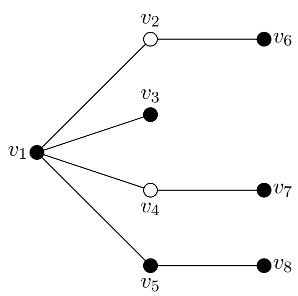

Figure 1: A graph $H \in \zeta$ with $A = \{v_1, v_2, v_3, v_4, v_5\}$ and $B = \{v_1, v_3, v_5, v_6, v_7, v_8\}$.

The next corollary shows that ζ is the set of all graphs G of order n satisfying $L_{2,t}(G) = n + 2 - \Delta(G)$.

Corollary 2.2. *Let G be a graph of order n, then, $L_{2,t}(G) \leq n+2-\Delta(G)$. Moreover, $L_{2,t}(G) = n+2-\Delta(G)$ if and only if $G \in \zeta$.*

Proof. Let S be an $L_{2,t}(G)$-set. Clearly, each component of $G[S]$ is a path, and $|N(v) \cap S| \leq 2$ for every vertex $v \in V(G)$. Let v' be a vertex of maximum degree in G. Similar to the proof of previous theorem, we have $L_{2,t}(G) = |S| = n - |\overline{S}| \leq n+2-\Delta(G)$.

Let now G be a graph of order n for which $L_{2,t}(G) = n+2-\Delta(G)$. It is easy to see that G has following properties:

(i) $|N[v'] \cap S| = 3$,

(ii) $V(G) \backslash N[v'] \subset S$.

Based on the above argument, we have $G \in \zeta$ with $N[v'] = A$ and $S = B$.

We now prove the converse. Assume that $G \in \zeta$. It suffices to show that $L_{2,t}(G) \geq n+2-\Delta(G)$. Let $A \cap B = \{u, u', u''\}$ and $|A| = a+1$, where v' is a vertex of degree a in $G[A]$. We claim that $\Delta(G) = a$. Every vertex in B has degree at most two in $G[B]$. So, each of the vertices u, u' and u'' is adjacent to at most two vertices in B. On the other side, each of u, u', u'' is adjacent to at most $a-2$ vertices in $A \backslash \{u, u', u''\}$. Thus, $deg(u) \leq a$, $deg(u') \leq a$ and $deg(u'') \leq a$. For each vertex $v_1 \in A \backslash \{u, u', u''\}$, v_1 is adjacent to at most $a-3$ vertices in $A \backslash \{u, u', u'', v_1\}$ and at most two vertices in B. So $deg(v_1) \leq a-1$ for every $v_1 \in A \backslash \{u, u', u''\}$. For each vertex $v_2 \in B \backslash \{u, u', u''\}$, v_2 is adjacent to at most $a-2$ vertices in $A \backslash \{u, u', u''\}$ and at most two vertices in B. Thus, $deg(v_2) \leq a$ for every $v_2 \in B \backslash \{u, u', u''\}$. Hence, $\Delta(G) \leq a$. But $deg(v') \geq a$, which implies that $\Delta(G) = a$. Note that B is a 2-TLP of G with $|B| = n - |A| + 3 = n+2-\Delta(G)$. Therefore, $L_{2,t}(G) \geq n+2-\Delta(G)$. So, this proof is complete. □

Corollary 2.3. *Let G be a r-regular graph of order n for which $L_{k,t}(G) = n+k-r$, where $k \leq r-1$. Then, $r \geq \frac{n+1}{2}$.*

Proof. If $r = n-1$, then G is a complete graph with $L_{k,t}(G) = k+1$ for $1 \leq k \geq n-2$. Hence, the result is true because $r = n-1 \geq \frac{n+1}{2}$. So we assume that $r \leq n-2$. Let S be an $L_{k,t}(G)$-set with $|S| = n+k-r$, and let $v \in V(G)$. Since $|N(v) \cap S| \leq k$, it follows that $|N(v) \cap \overline{S}| \geq r-k$. Clearly, $|\overline{S}| = n - |S| = r-k$. Thus, there exist exactly $r-k$ vertices, namely $v_1, v_2, \cdots, v_{r-k}$, in $N(v) \cap \overline{S}$. Inaddition, $\overline{S} = \{v_1, v_2, \cdots, v_{r-k}\}$. Let $U = V(G) \backslash N(v)$. Since $r < n-2$, it follows that $U \neq \phi$. Obviously, $U \subseteq S$. If $u \in U$, then $|N(u) \cap S| \leq k$. So every vertex $u \in U$ is adjacent to all vertices in $\overline{S}$, i.e. each vertex $v_i \in \overline{S}$ is adjacent to all $n-r$ vertices in U. notice that $deg(v_i) = r$ and v_i has at least one neighbor in $N[v]$. Thus, $n-r+1 \leq r$ and we have $r \geq \frac{n+1}{2}$. □

It is known that for any tree T, $\delta(T) = 1$. We denote the minimum degree of graph G taken over all non-leaf vertices by $\delta'(T)$.

Theorem 2.4. *Let $c \geq 4$ be a positive integer and let T be a tree of order n with $\delta'(T) \geq c$. Then, $L_{2,t}(T) \leq \frac{c-2}{c-1}n - c + 4$.*

Proof. We prove this theorem by induction on the order T. We have $n \geq c+1$, because $\delta'(T) \geq c$. If n equal to $c+1, c+2, \cdots, 2c-1$, then T is the star graph $K_{1,c}, K_{1,c+1}, \cdots, K_{1,2c-2}$, respectively. Thus, $L_{2,t}(T) = 3 \leq \frac{c-2}{c-1}n - c + 4$. Suppose that for all tree T' of order $n' < n$ with $\delta'(T') \geq c$, we have $L_{2,t}(T') \leq \frac{c-2}{c-1}n' - c + 4$.

Let now T be a tree of order n with $\delta'(T) \geq c$ and let S be an $L_{2,t}(T)$-set. We root T at r. Assume v' is a leaf of T at the furthest distance from r, and v'' is the parent of v'. Let L be the set of all leaves in $N(v'')$. We have $|L| \geq c-1$, because v'' is adjacent to at least $c-1$ leaves. Suppose T'' be obtained from T by deleting all the vertices of L. By induction, $L_{2,t}(T'') \leq \frac{c-2}{c-1}|V(T'')|-c+4 \leq \frac{c-2}{c-1}(n-(c-1))-c+4 = \frac{c-2}{c-1}n-2c+6$.

On the other hand, $|L \cap S| \leq |N[(v'') \cap S| \leq 2$. Therefore, $L_{2,t}(T) \leq L_{2,t}(T'')+2 \leq \frac{c-2}{c-1}n-2c+8 \leq \frac{c-2}{c-1}n-c+4$. □

If $diam(G)=1$, then G is a complete graph, and we know that $L_{2,t}(K_n)=2$. What can be said about the 2-total limited packing number of graphs with diameter 2. The following theorem is the answer of this question.

Theorem 2.5. *Let $c \geq 3$ be a positive integer. Then, there exists a graph G with $diam(G)=2$ for which $L_{2,t}(G)=c$.*

Proof. In what follows, we construct a graph G diameter 2 so that $L_{2,t}(G)=c$. Assume that $A=\{v_1,v_2,\cdots,v_c\}$ and $B=\{u_1,u_2,\cdots,u_{\frac{c(c-1)}{2}}\}$ with $A \cap B=\phi$. Let G be a graph with vertex set $V(G)=A \cup B$ so that $G[A]=cK_1$ and $G[B]=K_{\frac{c(c-1)}{2}}$ and each pair of distinct vertices in A has common neighbor in B. Clearly, $diam(G)=2$. It remains to see that $L_{2,t}(G)=c$. We have $|V(G)|=c+\frac{c(c-1)}{2}$ and $\Delta(G)=\frac{c(c-1)}{2}+1$. Hence, by Corollary 2.2, $L_{2,t}(G) \leq |V(G)|+2-\Delta(G)=c+1$. But $G \notin \zeta$, so $L_{2,t}(G) \leq c$.

On the other hand, A ia a 2-total limited packing of G. Therefore, $L_{2,t}(G)=c$. □

Proposition 2.6. *Let G be a graph without isolated vertex such that $\Delta(G) \geq 2$. Then,*

$$L_{1,t}(G)+1 \leq L_{2,t}(G) \leq \frac{\Delta(G)^2+1}{\delta(G)}L_{1,t}(G).$$

Proof. The lower bound is true for $\Delta(G) \geq 2$ [7]. We now verify the upper bound. Let $v \in G$ be an arbitrary vertex, then the set of vertices at distance at most two from v has at most $\Delta(G)^2+1$ vertices. Hence, $L_{1,t}(G) \geq \frac{2n}{\Delta(G)^2+1}$, by greedy algorithm.

Furthermore, $L_{k,t}(G) \leq \frac{kn}{\delta(G)}$ [7]. So, we have

$$L_{1,t}(G) \geq \frac{2n}{\Delta(G)^2+1} = \frac{2n\delta(G)}{(\Delta(G)^2+1)\delta(G)} \geq L_{2,t}(G)\frac{\delta(G)}{\Delta(G)^2+1}.$$

Therefore,

$$L_{2,t}(G) \leq \frac{\Delta(G)^2+1}{\delta(G)}L_{1,t}(G).$$

□

We can improve the above result for trees, as follows.

Theorem 2.7. *Let T be a given tree with $\Delta(T) \geq 2$, then*

$$L_{1,t}(T)+1 \leq L_{2,t}(T) \leq 2L_{1,t}(T).$$

Furthermore, both the following hold:

(i) $L_{1,t}(T)+1 = L_{2,t}(T)$ if and only if T is a star,

(ii) $L_{2,t}(T) = 2L_{1,t}(T)$ *if and only if for any* $L_{2,t}(T)$*-set* S *and any* $\gamma_t(T)$*-set* D *we have* $|N(s) \cap D| = 1$ *and* $|N(d) \cap S| = 2$ *for every* $s \in S$ *and every* $d \in D$*.*

Proof. We have proved this theorem by contradiction. □

We end with the following theorem and two examples that illustrate it.

Theorem 2.8. *Let* $a \geq 2$ *and* b *be two integers sush that* $a + 1 \leq b \leq 2a$*. Then, there exists a tree* T *for which* $L_{1,t}(T) = a$ *and* $L_{2,t}(T) = b$*.*

Example 2.9. Let $a = 4$ and $b = 7$. Consider a star $K_{1,4}$ with vertex set $\{r, v_1, v_2, v_3, v_4\}$ and $deg(r) = 4$. Let T be the tree obtained from $K_{1,4}$ by adding two leaves u_i and u'_i to each v_i for $1 \leq i \leq 2$ and one leaf u_3 to v_3. Figure 2 depicts T.

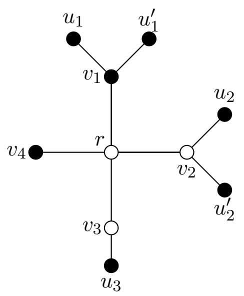

Figure 2: graph T

It is easy to see that $\{u_1, u'_1, u_2, u'_2, u_3, v_1, v_4\}$ is an $L_{2,t}(T)$-set. On the other hand, $L_{1,t}(T) = 4$.

Example 2.10. Assume now that $a = 5$ and $b = 10$. Let $P = v_1 v_2 \cdots v_5$ be a path. We add two leaves u_{i_1} and u_{i_2} to each v_i, and obtain tree T. Observe that $\{u_{1_1}, u_{2_1}, \cdots, u_{a_1}\}$ is a $L_{1,t}(T)$-set of T. Moreover, $\{u_{1_1}, u_{1_2}, u_{2_1}, u_{2_2} \cdots, u_{5_1}, u_{5_2}\}$ is a $L_{2,t}(T)$-set of T.

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