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Some inequalities for the exponential Zagreb indices

Mahdieh Azari
Department of Mathematics, Kazerun Branch, Islamic Azad University, P. O. Box: 73135-168, Kazerun, Iran

Farzaneh Falahati-Nezhad[1]
Department of Mathematics, Safadasht Branch, Islamic Azad University, Tehran, Iran

Nasrin Dehgardi
Department of Mathematics and Computer Science, Sirjan University of Technology, Sirjan, Iran

Abstract

The first and second Zagreb indices are among the best-known and most thoroughly investigated vertex-degree-based topological indices in mathematical chemistry. The exponential of these two invariants were proposed by Rada (MATCH Commun. Math. Comput. Chem. 82(1) 29–41) in 2019. In this paper, we present some inequalities for the exponential Zagreb indices of some classical graph products which connect them to the exponential Zagreb indices of their building blocks.

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1 Introduction

Let $\mathcal{G}$ be a nontrivial simple connected graph with vertex set $V_\mathcal{G}$ and edge set $E_\mathcal{G}$. We denote by $n_\mathcal{G} = |V_\mathcal{G}|$ and $m_\mathcal{G} = |E_\mathcal{G}|$ the order and size of $\mathcal{G}$, respectively. For a vertex $a \in V_\mathcal{G}$, the degree $d_\mathcal{G}(a)$ is the number of all vertices of $\mathcal{G}$ adjacent with a. If $d_\mathcal{G}(a) = d_\mathcal{G}(b)$ for each $a, b \in V_\mathcal{G}$, then $\mathcal{G}$ is called a *regular* graph. We denote by $\delta_\mathcal{G}$, the minimum degree $\mathcal{G}$.

Topological indices are real numbers related to the molecular graph of a chemical structure which are invariant under graph isomorphism. They are applied in predicting the physico-chemical properties of chemical structures and considered as helpful tools in QSPR/QSAR investigations (see, for example, [5]).

The *first and second Zagreb indices* are among the foremost and well-investigated vertex-degree-based topological indices. These indices were proposed by Gutman *et al.* [6, 8] in 1970s and formulated for a graph $\mathcal{G}$ as

$$M_1(\mathcal{G}) = \sum_{a \in V(\mathcal{G})} d_\mathcal{G}^2(a) \quad \text{and} \quad M_2(\mathcal{G}) = \sum_{aa' \in E(\mathcal{G})} d_\mathcal{G}(a) d_\mathcal{G}(a').$$

The readers interested in more information on the theory, applications and variants of Zagreb indices can be referred to the survey [7] and the references cited therein.

[1]Speaker

In 2019, Rada [10] put forward the *exponential* of a vertex-degree-based invariant. In particular, the *exponential of Zagreb indices* are defined as

$$e^{M_1}(\mathcal{G}) = \sum_{a \in V(\mathcal{G})} e^{d^2_{\mathcal{G}}(a)} \quad \text{and} \quad e^{M_2}(\mathcal{G}) = \sum_{aa' \in E(\mathcal{G})} e^{d_{\mathcal{G}}(a) d_{\mathcal{G}}(a')}.$$

For more information on the exponential Zagreb indices, see [3, 4, 11].

In this paper, we give lower bounds on the exponential first and second Zagreb indices of some classical graph products such as join, corona product, Cartesian product and composition in terms of the exponential Zagreb indices and some parameters of their components. We refer the interested readers to [1, 2, 9] for more information on topological indices of graph operations.

2 Main results

In this section, the exponential Zagreb indices are studied for certain graph products. The factors of each graph product are assumed to be simple connected graphs $\mathcal{G}$ and $\mathcal{H}$. We start with the join operation.

2.1 Join

Definition 2.1. The *join* $\mathcal{G} \nabla \mathcal{H}$ is the graph union $\mathcal{G} \cup \mathcal{H}$ together with all the edges joining $V_{\mathcal{G}}$ and $V_{\mathcal{H}}$.

Obviously, $n_{\mathcal{G}\nabla\mathcal{H}} = n_{\mathcal{G}} + n_{\mathcal{H}}$ and $m_{\mathcal{G}\nabla\mathcal{H}} = m_{\mathcal{G}} + m_{\mathcal{H}} + n_{\mathcal{G}} n_{\mathcal{H}}$.

Theorem 2.2. *The following inequalities hold.*

$$\begin{aligned} e^{M_1}(\mathcal{G}\nabla\mathcal{H}) &\geq e^{2n_{\mathcal{H}}\delta_{\mathcal{G}} + n^2_{\mathcal{H}}}\ e^{M_1}(\mathcal{G}) + e^{2n_{\mathcal{G}}\delta_{\mathcal{H}} + n^2_{\mathcal{G}}}\ e^{M_1}(\mathcal{H}), \\ e^{M_2}(\mathcal{G}\nabla\mathcal{H}) &\geq e^{2n_{\mathcal{H}}\delta_{\mathcal{G}} + n^2_{\mathcal{H}}}\ e^{M_2}(\mathcal{G}) + e^{2n_{\mathcal{G}}\delta_{\mathcal{H}} + n^2_{\mathcal{G}}}\ e^{M_2}(\mathcal{H}) + n_{\mathcal{G}} n_{\mathcal{H}}\ e^{(\delta_{\mathcal{G}} + n_{\mathcal{H}})(\delta_{\mathcal{H}} + n_{\mathcal{G}})}. \end{aligned}$$

The equality holds in both cases if and only if $\mathcal{G}$ and $\mathcal{H}$ are regular graphs.

2.2 Corona product

Definition 2.3. The *corona product* $\mathcal{G} \circ \mathcal{H}$ is the graph obtained by a copy of $\mathcal{G}$ and $n_{\mathcal{G}}$ copies of $\mathcal{H}$ and joining the ith vertex of $\mathcal{G}$ to each vertex in the ith copy of $\mathcal{H}$ for each $1 \leq i \leq n_{\mathcal{G}}$.

Obviously, $n_{\mathcal{G}\circ\mathcal{H}} = n_{\mathcal{G}} + n_{\mathcal{G}} n_{\mathcal{H}}$ and $m_{\mathcal{G}\circ\mathcal{H}} = m_{\mathcal{G}} + n_{\mathcal{G}} m_{\mathcal{H}} + n_{\mathcal{G}} n_{\mathcal{H}}$.

Theorem 2.4. *The following inequalities hold.*

$$\begin{aligned} e^{M_1}(\mathcal{G}\circ\mathcal{H}) &\geq e^{2n_{\mathcal{H}}\delta_{\mathcal{G}} + n^2_{\mathcal{H}}}\ e^{M_1}(\mathcal{G})\ + n_{\mathcal{G}}\ e^{2\delta_{\mathcal{H}} + 1}\ e^{M_1}(\mathcal{H}), \\ e^{M_2}(\mathcal{G}\circ\mathcal{H}) &\geq e^{2n_{\mathcal{H}}\delta_{\mathcal{G}} + n^2_{\mathcal{H}}}\ e^{M_2}(\mathcal{G}) + n_{\mathcal{G}}\ e^{2\delta_{\mathcal{H}} + 1}\ e^{M_2}(\mathcal{H}) + n_{\mathcal{G}} n_{\mathcal{H}} e^{(\delta_{\mathcal{G}} + n_{\mathcal{H}})(\delta_{\mathcal{H}} + 1)}. \end{aligned}$$

The equality holds in both cases if and only $\mathcal{G}$ and $\mathcal{H}$ are regular graphs.

2.3 Cartesian product

Definition 2.5. The *Cartesian product* $\mathcal{G}\square\mathcal{H}$ is a graph with vertex set $V_{\mathcal{G}\square\mathcal{H}} = V_{\mathcal{G}} \times V_{\mathcal{H}}$ and $(a,b)(a',b') \in E_{\mathcal{G}\square\mathcal{H}}$ if and only if ($a = a'$ and $bb' \in E_{\mathcal{H}}$) or ($b = b'$ and $aa' \in E_{\mathcal{G}}$).

Obviously, $n_{\mathcal{G}\square\mathcal{H}} = n_{\mathcal{G}}n_{\mathcal{H}}$ and $m_{\mathcal{G}\square\mathcal{H}} = n_{\mathcal{G}}m_{\mathcal{H}} + n_{\mathcal{H}}m_{\mathcal{G}}$.

Theorem 2.6. *The following inequalities hold.*

$$\begin{aligned} e^{M_1}(\mathcal{G}\square\mathcal{H}) &\geq e^{2\delta_{\mathcal{G}}\delta_{\mathcal{H}}}\ e^{M_1}(\mathcal{G})\ e^{M_1}(\mathcal{H}), \\ e^{M_2}(\mathcal{G}\square\mathcal{H}) &\geq e^{2\delta_{\mathcal{G}}\delta_{\mathcal{H}}}\big(e^{M_1}(\mathcal{H})e^{M_2}(\mathcal{G}) + e^{M_1}(\mathcal{G})e^{M_2}(\mathcal{H})\big). \end{aligned}$$

The equality holds in both cases if and only if $\mathcal{G}$ and $\mathcal{H}$ are regular graphs.

2.4 Composition

Definition 2.7. The *composition* $\mathcal{G}[\mathcal{H}]$ is a graph with vertex set $V_{\mathcal{G}[\mathcal{H}]} = V_{\mathcal{G}} \times V_{\mathcal{H}}$ and $(a,b)(a',b') \in E_{\mathcal{G}[\mathcal{H}]}$ if and only if $aa' \in E_{\mathcal{G}}$ or ($a = a'$ and $bb' \in E_{\mathcal{H}}$).

Obviously, $n_{\mathcal{G}[\mathcal{H}]} = n_{\mathcal{G}}n_{\mathcal{H}}$ and $m_{\mathcal{G}[\mathcal{H}]} = m_{\mathcal{G}}n_{\mathcal{H}}^2 + n_{\mathcal{G}}m_{\mathcal{H}}$.

Theorem 2.8. *The following inequalities hold.*

$$\begin{aligned} e^{M_1}(\mathcal{G}[\mathcal{H}]) &\geq n_{\mathcal{G}}\ e^{n_{\mathcal{H}}^2\delta_{\mathcal{G}}^2 + 2n_{\mathcal{H}}\delta_{\mathcal{G}}\delta_{\mathcal{H}}}\ e^{M_1}(\mathcal{H}), \\ e^{M_2}(\mathcal{G}[\mathcal{H}]) &\geq n_{\mathcal{H}}^2\ m_{\mathcal{G}}\ e^{n_{\mathcal{H}}^2\delta_{\mathcal{G}}^2 + 2n_{\mathcal{H}}\delta_{\mathcal{G}}\delta_{\mathcal{H}} + \delta_{\mathcal{H}}^2} + n_{\mathcal{G}}\ e^{n_{\mathcal{H}}^2\delta_{\mathcal{G}}^2 + 2n_{\mathcal{H}}\delta_{\mathcal{G}}\delta_{\mathcal{H}}}\ e^{M_2}(\mathcal{H}). \end{aligned}$$

The equality holds in both cases if and only if $\mathcal{G}$ and $\mathcal{H}$ are regular graphs.

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e-mail: mahdie.azari@gmail.com, mahdieh.azari@iau.ac.ir
e-mail: farzanehfalahati_n@yahoo.com
e-mail: n.dehgardi@sirjantech.ac.ir