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Improved upper bounds for Lee metric codes

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Abstract

Lee metric codes are important objects both in theory and application. In this paper, we employ a concatenation coding scheme to construct binary codes from Lee metric codes. As a consequence of this method, we are able to improve the best known upper bound for Lee metric codes of certain parameters from the work of [2].

Keywords: Lee metric codes, Error correction codes, Concatenation codes, Plotkin bound

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1 Introduction

Lee metric code was introduced in [1]. Lee metric is particularly suitable for schemes that modulate the signal by changing its phase. It has also been applied to correct burst errors in multiple dimensions, channels that have constraints or partial responses, schemes that use interleaving, and error correction for flash memories.

Let Q be an alphabet of size $|Q| = q$. A code C of length n is a subset $C \subseteq Q^n$. When $q = 2$, we refer to the code as binary. The elements of C are called codewords. The minimum distance of a code C is the minimum of the distances between two different codewords, and denoted as d. Note that in the definition of the minimum distance of a code, we need some specific measure of distance. The Hamming distance of two vectors is defined as the number of coordinates where they differ.

In Lee distance, the underlying alphabet is the set of integers modulo a natural number q. The distance between two vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ over the this alphabet is equal to $\sum_{i=1}^{n} \min\{|x_i - y_i|, q - |x_i - y_i|\}$ where the subtraction is taken modulo q.

Also, $A_q^L(n, d)$ denotes the maximum size of Lee codes of length n over the alphabet $\mathbb{Z}_q$, and the minimum distance of d. Similar to this, $A_2(n, d)$ is the maximum size of binary codes of length n and the minimum distance d.

[1] speaker

The Plotkin bound states that if $2d > n$ then $A_2(n,d) \leq \frac{2d}{2d-n}$. We follow the notations and terminologies of [5] and [6].

A Lee metric code of parameters (n,q,d) can be considered as a collection of n-length vectors of the vertices of the cycle graph C_q such that any pair of the vectors in the family are of distance at least d. The notion of distance in the context of Lee metric code is defined as the sum of the distances between the entries of the vector, as the vertices of C_q. In other words, every pair of the vertices of the cycle graph C_q has a distance equal to the length of the shortest path connecting them in C_q. Now, when two vectors whose coordinates are vertices of C_q are given, the Lee distance of them is the summation of the distances of their corresponding entries. For instance, when $q = 3$, the distance of any pair of distinct vertices of C_3 is equal to 1. Thus, the induced Lee distance is in fact Hamming distance.

A basic question regarding the Lee metric code is to find the upper and lower bounds for $A_q^L(n,d)$. In this paper, we introduce a concatenation scheme for converting any Lee metric code to a code over finite fields and in particular over the binary field. As a consequence of this transformation, we find new upper bounds for the maximum size of a Lee metric code for certain sets of parameters.

2 Main results

We aim to find a function $f : V(C_q) \to \{0,1\}^m$ with the following property. For every pair $u, v \in V(C_q)$, their graph distance (Lee distance) $d_G(u,v)$ is less than or equal to the Hamming distance $d_H(f(u), f(v))$. In the next lemma, we will find the smallest value of m for cycle graphs.

Lemma 2.1. *Consider the cycle graph C_q. Let m be the smallest integer number such that there exists a function $f : V(C_q) \to \{0,1\}^m$ with the property that $\forall u, v \in V(C_q) : d_G(u,v) \leq d_H(f(u), f(v))$. Then, $m = \lceil \frac{q}{2} \rceil$.*

Proof. Consider two vertices v and u of distance $\lceil \frac{q}{2} \rceil$. Since the distance of any pair of the vertices is a lower bound on m, we must have $m \geq \lceil \frac{q}{2} \rceil$. We will show a way to assign binary codes of length $\lceil \frac{q}{2} \rceil$ to the q vertices of the cycle such that the condition of the function f is satisfied. We can start by assigning the code of all zeros to one vertex. Then, we can add ones to the right and left ends of the code and assign them to the adjacent vertices and so forth. The following figure illustrates this method for C_6. □

The next lemma is an analogous result when the edges of the cycle have the same integer weight.

Lemma 2.2. *Consider a cycle C_q with the constant weight w on each edge. The minimum m for which a function $f : V(C_q) \to \{0,1\}^m$ exists such that $\forall u, v \in V(C_q) : d_G(u,v) \leq d_H(f(u), f(v))$ is $\lceil \frac{qw}{2} \rceil$.*

Proof. The proof is similar to that of the previous. □

Now, suppose that a Lee metric code over the cycle C_q of parameters (n,d) is given. We convert each codeword of the Lee metric code to a binary codeword as follows. For every codeword $(v_1, \ldots, v_n)$, we consider the binary vector $(f(v_1), \ldots, f(v_n))$ in which f has the property that $d_G(u,v) \leq d_H(f(u), f(v))$. Here, we assumed that all the edges of C_q have weight w.

Note that if $f(v_i)$'s are of length m, then each vector of length n of the vertices of C_q is converted to a binary vector of length exactly $m \cdot n$. Furthermore, because of the definition of f, if two codewords of the Lee metric code are of distance t, after the transformation, the distance of the transformed codewords is at least $w \cdot t$.

This simple observation implies the following theorem which relates $A_2(n, d)$ and $A_q^L(n', d)$.

Theorem 2.3. *Consider a simple cycle graph C_q, and $n, d \in \mathbb{N}$. We have $A_q^L(n, d) \leq A_2(n\lceil \frac{q}{2} \rceil, d)$.*

Example 2.4. For $n = 2$ and $d = 3$, the maximum size of a Lee metric code over C_6 is 4. (See tables in [2].) Those codes on a cycle with vertices of $\{1, 2, 3, 4, 5, 6\}$ respectively around the cycle can be:

$$11, 14, 41, 44$$

By 1, we know binary code for the vertex 1 is 000 and for the vertex 4, is 111. Therefore, by concatenating method, the binary codes will be

$$000000, 000111, 111000, 111111$$

which have length 6 and minimum distance of 3.

Corollary 2.5. *For weighted cycle C_q with constant weight of w, we have: $A_q^L(n, d) \leq A_2(n\lceil \frac{wq}{2} \rceil, wd)$*

A direct implication of the theorem and its corollary is that any upper bound on the size of the binary code of appropriate parameters implies the same upper bound for the corresponding Lee metric code. In the continuation, we present two special cases in which we improve the best known upper bound for the Lee metric codes.

2.1 Improved upper bound for $A_6^L(8, 14)$

The lower and upper bounds of Lee metric codes have been investigated in many papers. Using the previous theorem, we can improve the upper bound for $A_6^L(8, 14)$, which is less than or equal to 7 by linear programming methods. (See [2], [3], [4].)
We show that the upper bound is 6. This is because there are exactly 6 binary codewords with length 24 and minimum distance 14. Therefore, we have $A_6^L(8, 14) \leq A_2(8 \times 3, 14) = 6$. This is because we have exactly 6 binary codewords with length 24 and a minimum distance of 14.

2.2 Improved upper bound for $A_{17}^L(4, 19)$

The best upper bound for Lee codes on C_{17} with length 4 and minimum distance 19 is 11 (See [2]). We can find the minimum of m in Lemma 2 for C_{17} with a constant weight of 2 on each edge. That length is 17. Therefore by Corollary 2, we conclude that $A_{17}^L(4, 19) \leq A_2(68, 38)$. On the other hand, by well-known Plotkin bound we know $A_2(68, 38) \leq 2\lfloor \frac{38}{38 \times 2 - 68} \rfloor = 8$.
Finally, we must mention that this method can be extended to any alphabet size.

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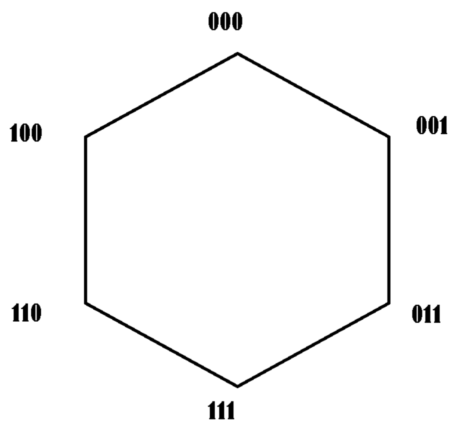

Figure 1: Binary codes on C_6