A characterization of commutative rings in which every semi co-Hopfian module is artinian

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Abstract

Let R be an associative ring with unit $1 \neq 0$, we call an unital left R-module M semi co-Hopfian (resp semi Hopfian) if any injective (resp. surjective) endomorphism of M has a direct summand image (resp kernel). Starting from that every artinian module is co-Hopfian and so semi co-Hopfian, we showad in this paper that the class of ring on which every semi co-Hopfian module is artinian coincide with the class of artinian principal ideal rings when the v.p is satisfied. Moreover, some properties of this class of rings are given.

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1 Introduction

The study of classes of rings (modules) by properties of their endomorphisms is a classical research subject since 1960s. In 1986, Hiremath [3] introduced the concept of Hopfian modules and rings. Later, the dual concept cohopfian modules and rings were given. Hopfian and co-Hopfian modules (rings) have been investigated by several authors. In 2008, P. AYDOGDU and A. C. OZCAN introduced and investigated the semi Hopfian and semi co-Hopfian modules.

Recall that a module M is called co-Hopfian (resp. Hopfian) if any injective (resp. surjective) endomorphism of M is an automorphism. Note that any artinian (resp noetherian) module is co-Hopfian (resp. Hopfian). A module M is called semi co-Hopfian (resp. semi Hopfian) if any injective (resp. surjective) endomorphism of M has a direct summand image (resp. kernel). In other words, any injective (resp. surjective) endomorphism of M splits.

Clearly, any co-Hopfian (resp. Hopfian) module is semi co-Hopfian (resp semi Hopfian). The converse is not true in general, for example let $\mathbb{Z}_M = \mathbb{Q}^{(\mathbb{N})}$. By [2] M is semi co-Hopfian but not co-Hopfian.

[1] speaker

In [2] M is semi co-Hopfian if and only if any submodule N of M which is isomorphic to M is a direct summand of M, therefore, the concept of semi co-Hopfian module is a generalization of co-Hopfian module. The aim of this paper is to characterize the rings R, on which any semi co-Hopfian module is artinian. These rings are named $SCHA$-rings.
Throughout this paper, R denotes an associative ring with identity $1 \neq 0$, and modules M are unitary left R-modules. The property $Hom(M_i, M_j) = Hom(M_j, M_i) = 0$ whenever $i \neq j$ for a family $\{M_i\}$ of R-modules, is named Vanaja property (briefly v.p).
Let R be a ring. A module M is noetherian (resp. artinian) if any ascending (resp. descending) chain of submodules of M stabilizes. M is strongly co-Hopfian (resp. strongly Hopfian) if for any endomorphism f of M the descending chain $Im(f) \supseteq Im(f^2) \supseteq \cdots$ (resp. ascending chain $Ker(f) \subseteq Ker(f^2) \subseteq \cdots$) stabilizes. M is called Dedekind finite if $M = M \oplus N$ for some module N, $N = 0$. The socle of M ($Soc(M)$) is defined to be the sum of the minimal nonzero submodules of M. A submodule of M is essential if it has a non-trival intersection with every non-trivial submodule of M: that is, $E \cap L = 0$ implies $L = 0$ for a submodule L of M. M is finitely cogenerated if only if $Soc(M)$ is essential and finitely generated.

2 Preliminary results

Lemma 2.1. *The following are equivalent for a module M.*

1. *M is co-Hopfian*
2. *M is Dedekind finite and semi co-Hopfian*
3. *M is weakly co-Hopfian and semi co-Hopfian.*

Proof. (3) $\Leftrightarrow$ (1) $\Rightarrow$ (2) obvious
(2) $\Rightarrow$ (1) Let f be an injective endomorphism of M. Then $f(M) \oplus K$ for $K \leqslant M$. Define a homomorphism $\varphi : M \oplus K \longrightarrow M$ by $\varphi(m, k) = f(m) + k$. Then φ is an isomorphism. Since M is Dedekind finite, $K = 0$. Hence $f(M) = M$ and so f is an isomorphism. □

Lemma 2.2. *Any direct summand of semi co-Hopfian modules is semi co-Hopfian.*

Proof. Let N be a direct summand of M and $f : N \longrightarrow N$ a monomorphism. Write $M = N \oplus N'$. Then $g : M \longrightarrow M$, $g(n + n') = f(n) + n'$ where $n \in N$, $n' \in N'$, is a monomorphism. Since $Im(g) = Im(f) \oplus N'$ is a direct summand of M, we get that $Im(f)$ is a direct summand of N. □

Lemma 2.3. *(Theorem .0.1. of [4])*

1. *Let M be a co-Hopfian (Hopfian) module. If M decomposes as a direct sum of a family $\{M_i\}$ of nontrivial R-modules, then each M_i is co-Hopfian (Hopfian).*
2. *Let $\{M_i\}$ be a family of family of nontrivial R-modules. We suppose that v.p is satisfied. If each M_i is co-Hopfian (Hopfian), then so is M.*

Lemma 2.4. *(Proposition 10.18 of [1])*
For each ring R, the following statements are equivalent:

1. *R is left artinian;*

2. *Every finitely generated left R-module is finitely cogenerated.*

Remark 2.5. :

1. Every cyclic module is finitely generated.

2. Any co-Hopfian module is semi co-Hopfian. But the converse is not true in general. For example, let ${}_{\mathbb{Z}}M = \mathbb{Q}^{(\mathbb{N})}$. Since M is quasi-injective, it is semi co-Hopfian. But $M \cong M \oplus \mathbb{Q}$ is not Dedekind finite, hence not co-Hopfian.

Lemma 2.6. *Every finitely cogenerated module M is Dedekind finite.*

Proof. At beginning recall that if $Soc(M)$ is essential and Dedekind finite, then M is Dedekind finite. M finitely cogenerated implies $Soc(M)$ is essential. Now let's prove that $Soc(M)$ is Dedekind finite. By definition $Soc(M)$ is a direct sum of all simple submodules of M, hence $Soc(M)$ is a semisimple submodule of M. We can write $Soc(M) = \bigoplus_{i\in I} S_i$. Since $Soc(M)$ is finitely generated then I is finite and so $Soc(M)$ is of finite lenght. Therefore $Soc(M)$ is Dedekind finite. □

An I-ring (S-ring) is a ring such that every co-Hopfian (Hopfian) module is artinian (noetherian).

Lemma 2.7. *(Theorem 9 of [5])*
For a commutative ring R, the following are equivalent:

1. *R is a artinian principal ideal ring;*

2. *R an I-ring;*

3. *R is S-ring.*

Lemma 2.8. *(Lemma 2 of [5] p.247)*
Every integral domain S-ring is a field.

3 Aim results

Proposition 3.1. *Let R be a commutative ring. If R is a $SCHA$-ring, then R is an I-ring.*

Proof. Assume that R is a $SCHA$-ring. Let M be a co-Hopfian module, then by second point of remark 2.5, M is semi co-Hopfian. Therefore M is artinian. □

Theorem 3.2. *Let R be a commutative ring. We suppose that v.p is satisfied. The following are equivalent:*

1. *R is a artinian principal ideal ring;*

2. *R is $SCHA$-ring.*

Proof. (2) ⇒ (1) Results from Lemma 2.6 and proposition 3.1.
(1) ⇒ (2) Let M be a semi co-Hopfian module. R principal ideal ring implies every R-module is a direct sum of cyclic modules. Let $M = \bigoplus M_i$. We can write $M = M_i \oplus (\oplus_{i\neq j}(M_j))$, in fact every M_i is a direct summand of M. By lemma 2.2, every M_i is semi co-Hopfian.($\star$)
Recall every M_i is cyclic hence finitely generated. Since R artinian, referring to lemma 2.4, for each i, M_i

is finitely cogenerated. It results by lemma 2.6 for each i M_i is Dedekind finite. $(\star\star)$.
$(\star)$ and $(\star\star)$ imply for each i , M_i is Dedekind finite and semi co-Hopfian. By lemma 2.1 for each i, M_i is co-Hopfian. Referring to the second point of lemma 2.3 M is co-Hopfian. Over artinian principal ideal ring, co-Hopfian and artinian modules coincide. Therefore M is artinian.
Conclusion R is a $SCHA$-ring. □

Corollary 3.3. *For a commutative ring R, if v.p is satisfied then the following are equivalent:*

1. *R is an artinian principal ideal ring;*
2. *R is an I-ring;*
3. *R is a $SCHA$-ring.*
4. *R is a S-ring*

Proof. Results from theorem 9 of [5] and theorem 3.2. □

Proposition 3.4. *Let R be a commutative $SCHA$-ring. We suppose that v.p is satisfied. Then every prime ideal is maximal. Also, there are only finitely many prime ideals.*

Proof. Let P be a prime ideal of R, R/P is a commutative integral domain. By referring successively corollary 3.3 and lemma 2.8, R/P is a field. Therefore P is maximal.
For the second statement we denote by L the set of all prime ideals. Let $P \in L$, R/P is a field so R/P is simple. Furthemore if P and $P' \in L$ with P
$nsubseteqP'$. Since v.p is satisfied then $Hom(R/P, R/P') = 0$. Every simple module is semi co-Hopfian but also Dedekind finite. By lemma 2.1 R/P is co-Hopfian. If v.p is satisfied, direct sum of co-Hopfian modules is co-Hopfian. Therefore $M = \bigoplus_{P\in L} R/P$ is co-Hopfian. Over $SCHA$-ring every co-Hopfian module is artinian. Hence L is finite. □

Lemma 3.5. *(Theorem of [7]) Any injective endomorphism of a finitely generated R-module is an isomorphism if and only if every prime ideal of R is maximal.*
In other words all finitely generated R module is co-Hopfian if and only if all prime ideals of R are maximal.

Recall a ring R is strongly-π-regular if every cyclic R-module is strongly co-Hopfian.

Proposition 3.6. *Let R be a commutative $SCHA$-ring. We suppose that v.p is satisfied. Then R is strongly-π-regular.*

Proof. Let M be a cyclic R-module, then M is finitely generated. By lemma 3.5 and proposition 3.4, M is co-Hopfian. Since the class of $SCHA$-rings and the class of I-rings coincide if v.p is satisfied, then M is artinian. Every artinian ring is strongly co-Hopfian, therefore M is strongly co-Hopfian.
In conclusion R is strongly-π-regular. □

Proposition 3.7. *Let's suppose v.p is satisfied, A finite direct product $R = \prod R_i$ of $SCHA$-rings is a $SCHA$-ring if and only if every R_i is a $SCHA$-ring.*

Proof. First let's suppose $R = \prod R_i$ is a $SCHA$-ring and M_i a semi co-Hopfian R_i-module. From this surjective homomorphism $P_i : \prod R_i \longrightarrow R_i$ every R_i-module has a structure of R-module. Hence M_i is artinian.
Secondly, let's suppose every R_i is a $SCHA$-ring. By v.p we can write $M = \oplus M_i = M_i \oplus (\oplus_{i \neq j}(M_j))$ with each M_i is a R_i-module. M semi co-Hopfian by lemma 2.2 every M_i is semi co-Hopfian. Therefore every M_i is artinian. A finite direct product of artinian rings is artinian. In conclusion $R = \prod R_i$ is a $SCHA$-ring. □

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