Principally Supplemented Semimodules

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ABSTRACT

In this work, principally supplemented semimodules are defined which generalize supplemented semimodules. Some properties of principally supplemented semimodules are investigated. We show that if $A = A_1 \oplus A_2$ is duo semimodule with A_1 and A_2 are principally supplemented semimodules, then A is principally supplemented. It is also proved that if A is an indecomposable semimodule, then A is principally supplemented semimodule if and only if A is principally lifting semimodule.

KEYWORDS: Supplemented semimodules, Lifting semimodules, Principally supplemented semimodules, Principally lifting semimodules, Semiperfect semimodules.

1 INTRODUCTION

Throughout this article, S denotes a commutative semiring with identity and A will denote an unitary left S-semimodule. A (left) R-semimodule A (denoted by $_RA$) is a commutative additive semigroup which has a zero element 0_A, together with a mapping from $R \times A$ into A (sending (r,a) to ra) such that $(r+s)a = ra + sa$, $r(a+b) = ra + rb$, $r(sa) = (rs)a$ and $0a = r0_A = 0$ for all $a, b \in A$ and $r, s \in R$ [8]. We say that N is an S-subsemimodule of A, denoted by $N \leq A$, if and only if N is itself an S-semimodule. A subsemimodule $N \leq A$ is called small in A (we write $N \ll A$), if for each subsemimodule $K \leq A$, with $N + K = A$ implies that $K = A$ [13]. $Rad(A)$ is the sum of all small subsemimodules of A [13]. A semimodule A is called hollow, if all proper subsemimodules of A are small in A [5]. A is called simple if it does not have a nontrivial subsemimodules. Let $U, K \leq A$. R. Wisbauer [14] study in details the notion of supplemented modules. Here, we study on the notion of supplemented semimodules. A subsemimodule K is called a supplement of U in A if it is minimal with respect to $A = U + K$ [6, 9]. A subsemimodule K of A is a supplement (weak supplement) of U in A if and only if $A = U + K$ and $U \cap K \ll K$ ($L \cap K \ll A$). A semimodule A is called supplemented (weakly supplemented) if every subsemimodule U of A has a supplement (weak supplement) in A [6, 7]. A is called lifting (or has the condition (D_1)) if, for every subsemimodule $N \leq A$, there exists a decomposition $A = X \oplus Y$ such that $X \leq N$ and $N \cap Y$ is small in A [10, 11]. A subsemimodule $U \leq A$ is called a subtractive subsemimodule of A if $a, a+b \in U$ then $b \in U$ [3, 4, 8]. If every subsemimodule U of A is subtractive, then A is called subtractive semimodule. If K is a subtractive subsemimodule of A, then A/K is an S-semimodule [8, p.165]. In Section 2, we give properties of small subsemimodules. Section 3 is devoted to introducing the concept of principally supplemented semimodules. In Section 4, we study on principally semiperfect semimodules by using the notions of principally supplemented and principally lifting semimodules.

2 ON SMALL SUBSEMIMODULES

Let A be semimodule. In [13] and [5], a subsemimodule N of M is called a small (or superfluous) subsemimodule if, whenever $M = N + X$, we have $M = X$. Small subsemimodule is named superfluous subsemimodule in [13]. We begin by the following lemma.

Lemma 2.1. Let A be a semimodule. Then we have the following

(1) If $L \ll A$ and $f: A \longrightarrow U$ is a homomorphism, then $(L) \ll U$. In particular, if $L \ll A \leq U$, then $L \ll U$.

(2) Let N, K be subsemimodules of A with $K \ll A$ and $N \leq K$. Then $N \ll A$.

(3) Let A = $A_1 \oplus A_2$ and $L_1 \leq A_1 \leq A$, $L_2 \leq A_2 \leq A$. Then $L_1 \oplus L_2$ is small in $A_1 \oplus A_2$ if and only if L_1 is small in A_1 and L_2 is small in A_2 .

Proof. (1) and (2) in [6, Lemma 2.4]. The proof of (3) similar to the proof that of [1, Proposition 5.20]. □

Lemma 2.2. Let U and V be subsemimodules of A. Then the following are equivalent:

(1) $A = U + V$ and $U \cap V$ is small in V.

(2) $A = U + V$ and for any proper subsemimodule H of V, $A \neq U + H$.

Proof. (1) ⟹ (2) Let $F, H \leq A$ with $A = F + H$. Then $K = (K \cap F) + H$. As $K \cap F \ll K$, $K = H$.

(2) ⟹ (1) If $K = (F \cap K) + H$ where $H \leq K$, then $A = F + K = F + H$. By (2), $H = K$. Therefore $\cap K \ll K$. □

Definition 2.3 [7]. A semimodule A is called a distributive if for all subsemimodules H, K, and F, then $F \cap (H + K) = F \cap H + F \cap K$ or $F + (H \cap K) = (F + H) \cap (F + K)$.

Lemma 2.4: Let $A = A_1 \oplus A_2 = K + N$ and $K \leq A_1$. If A is distributive and $K \cap N \ll N$, then $K \cap N \ll A_1 \cap N$.

Proof. Let $A_1 \cap N = (K \cap N) + L$. Since A is distributive, $N = A_1 \cap N \oplus A_1 \cap N$. We have

$$A = K + N = K + A_1 \cap N + A_2 \cap N = K + L + (A_2 \cap N)$$

and

$N = K \cap N + L + (A_2 \cap N)$. Since $K \cap N \ll N$, $N = L \oplus (A_2 \cap N)$. This and

$N = (N \cap A_1) \oplus (N \cap A_2)$ and $L \leq A_1 \cap N$ imply $L = A_1 \cap N$. Thus $K \cap N \ll A_1 \cap N$. □

3 ON PRINCIPALLY SUPPLEMENTED (LIFTING) SEMIMODULES

In this section, we introduce and study the concepts of principally supplemented and principally lifting semimodules. Some properties of these semimodules are obtained. We give the following definitions similar to that of [2].

Definition 3.1. A semimodule A is called principally supplemented if every cyclic subsemimodule U of A, there is a subsemimodule V of A such that $A = U + V$ with $U \cap V \ll V$.

Definition 3.2. Let U be a cyclic subsemimodule of A. A subsemimodule V is said to be a principally supplement of U in A if U and V satisfy the conditions in Lemma 2.2, and the semimodule A is called be principally supplemented if every cyclic subsemimodule of A has a principally supplement in A.

Definition 3.3. A semimodule A is called principally lifting (or PD_1) if all cyclic subsemimodule N of A there exists a decomposition $A = H \oplus K$ with $H \leq N$ and $N \cap K \ll A$. A semimodule A is called principally lifting (or PD_1) if for all $a \in A$, A has a decomposition $A = N \oplus H$ with $N \leq Sa$ and $(Sa) \cap H \ll H$.

Remark 3.4. It is known, every supplemented semimodule and every lifting semimodule, hence every principally lifting semimodule is principally supplemented. Here principally supplemented semimodules but not supplemented and no principally lifting.

Example 3.5. (1) The $\mathbb{Z}$-semimodule $\mathbb{Q}$ of rational numbers has no maximal subsemimodules. Every cyclic subsemimodule of $\mathbb{Q}$ is small, hence $\mathbb{Q}$ is principally supplemented $\mathbb{Z}$-semimodule. But $\mathbb{Q}$ is not supplemented.

(2) Suppose the $\mathbb{Z}$-semimodule $A = \mathbb{Q} \oplus (\mathbb{Z}/2\mathbb{Z})$. We show A is principally supplemented semimodule but is not supplemented. Let $(n, \bar{t}\,) \in A$. We first show that $(n, \bar{t}\,)\mathbb{Z}$ has a supplement in A. We divide the proof into some cases:

Case I: $n = 1$ and $\bar{t} = 1$. It is rutin to prove that $A = (1\,, \bar{1}\,)\mathbb{Z} + (\mathbb{Q} \oplus (\bar{0}))$ with $(1\,, \bar{1}\,)\mathbb{Z} \cap (\mathbb{Q} \oplus (\bar{0})) = (1\,, \bar{0})\mathbb{Z} \ll (\mathbb{Q} \oplus (\bar{0}))$.

Case II: $n = 1$ with $\bar{t} = 0$. Then $(n\,, \bar{t}\,)\mathbb{Z} = (1\,, \bar{0})\mathbb{Z} \ll \mathbb{Q} \oplus (\bar{0}))$.

Case III: $n = 0$ with $\bar{t} = \bar{1}$. Then $(n, \bar{t}\,)\mathbb{Z} = (1, \bar{0})\mathbb{Z} \leq^{\oplus} A$.

Case IV: $n \neq 1, 0$ with $\bar{t} = \bar{1}$. Let $(a\,, \bar{b}) \in A$. We prove (a , $\bar{b}$) $\in (n, \bar{1})\mathbb{Z} + (\mathbb{Q} \oplus (\bar{0}))$. If $\bar{b} = \bar{1}$, then $(a\,, \bar{b}) = (a\,, \bar{1}) = (n\,, \bar{1}) + (a - n\,, \bar{0}) \in (n\,, \bar{1})\mathbb{Z} + (\mathbb{Q} \oplus (\bar{0}))$. Suppose that $\bar{y} = \bar{0}$. Then $(a\,, \bar{b}) = (a\,, 0) = (n\,, \bar{1})0 + (a\,, \bar{0}) \in (n\,, \bar{1})\mathbb{Z} + (\mathbb{Q} \oplus (\bar{0}))$. Therefore $(a\,, \bar{b}) \in (n\,, \bar{1})\mathbb{Z} + (\mathbb{Q} \oplus (\bar{0}))$ and thus $A = (n\,, \bar{1})\mathbb{Z} + (\mathbb{Q} \oplus (\bar{0}))$. Since $((n\,,\bar{1})\,\mathbb{Z}\,) \cap (\mathbb{Q} \oplus (\bar{0})) = (2n\,, \bar{0})\mathbb{Z}$ and $(2n\,, \bar{0})\mathbb{Z}$ is small in $\mathbb{Q} \oplus (\bar{0})$. As a result that, in which cases, $(n\,, \bar{t})\mathbb{Z}$ has a supplement in A and A is principally supplemented $\mathbb{Z}$-semimodule. If A is supplemented $\mathbb{Z}$-semimodule, its direct summand $\mathbb{Q}$ will be a supplemented $\mathbb{Z}$-semimodule. A contradiction. Therefore A is not supplemented.

(3) Suppose the $\mathbb{Z}$-semimodules $A_1 = \mathbb{Z}/2\mathbb{Z}$ and $A_2 = \mathbb{Z}/8\mathbb{Z}$. It is easy to know A_1 and A_2 are principally supplemented. Let $A = A_1 \oplus A_2$. Then A is principally supplemented semimodules $\mathbb{Z}$-semimodule but is not principally lifting. Take $U_1 = (\bar{1}, \bar{2})\mathbb{Z}$, $U_2 = (\bar{1}, \bar{1})\mathbb{Z}$, $U_3 = (\bar{0}, \bar{2})\mathbb{Z}$, $U_4 = (\bar{0}, \bar{4})\mathbb{Z}$, $U_5 = (\bar{1}, \bar{4})\mathbb{Z}$, A_1 with A_2 are proper cyclic subsemimodules of A. $A = A_1 \oplus A_2 = U_2 \oplus U_5$ with U_3, U_4 are small subsemimodules of A. $A = U_1 + U_2$ with $U_1 \cap U_2 = U_4 \ll U_2$. Therefore A is principally supplemented semimodule. Since $A = U_1 + U_2$, U_1 is not small in A and it is not a direct summand of A and without any nonzero values direct summand of A. Therefore A is not principally lifting.

Let A be a semimodule. A subsemimodule N is called fully invariant if for each endomorphism f of A, $f(N) \leq N$. The left S-semimodule A is called a duo semimodule provided every subsemimodule of A is fully invariant.

Lemma 3.6. Let a subtractive semimodule $A = \bigoplus_{i \in I} A_i$ be a direct sum of subsemimodules $A_i (i \in I)$ and let U be a fully invariant subsemimodule of A. Then $U = \bigoplus_{i \in I} (U \cap A_i)$.

Proof. For any $j \in I$, let $P_j : A \longrightarrow A_j$ use the canonical projection as a symbol and let $i_j : A_j \longrightarrow A$ denote inclusion. Then $i_j\, p_j$ is an endomorphism of A and therefore $i_j\, p_j\ (U) \subseteq U$ for each $j \in I$. Thus, it follows $U \subseteq \bigoplus_{j \in I}\ i_j\, p_j\ (U) \subseteq \bigoplus_{j \in I} (U \cap A_j) \subseteq U$, so that $U = \bigoplus_{j \in I} (U \cap A_j)$. □

Remark 3.7. Finite direct sum of supplemented semimodules is again supplemented. But this is not the case for principally supplemented semimodules. But it is the case for some classes of semimodules.

The following three theorems can be thought of as a generalization of [2, Theorems 9, 10 and 11].

Theorem 3.8. If $A = A_1 \oplus A_2$ is a decomposition of A with A_1 and A_2 are principally supplemented semimodules. If A is a duo semimodule, then A is principally supplemented.

Proof. Let $A = A_1 \oplus A_2$ be a semimodule and Sa be a cyclic fully invariant subsemimodule of A. By Lemma 3.6, $Sa = ((Sa) \cap A_1) \oplus ((Sa) \cap A_2)$. Let $a = a_1 + a_2$ where $a_1 \in A_1$, $a_2 \in A_2$. Then Sa_1 = $(Sa) \cap A_1$ and Sa_2 = $(Sa) \cap A_2$. Since $(Sa) \cap A_1$ and $(Sa) \cap A_2$ are principal subsemimodules of A_1 and A_2 respectively, there is $N_1 \leq A_1$ such that $A_1 = a_1 + N_1$, $(Sa_1) \cap N_1 \ll N_1$ and $N_2 \leq A_2$ such that A_2 = $(Sa_2) + N_2$ and $(Sa_2) \cap N_2 \ll N_2$. Then A = $(Sa_1) + (Sa_2) + N_1 + N_2$ = $(Sa) + N_1 + N_2$. We prove $(Sa) \cap (N_1 + N_2) \ll N_1 + N_2$.

$$\begin{aligned}(Sa) \cap (N_1 + N_2) &= ((Sa) \cap A_1 + (Sa) \cap A_2) \cap (N_1 + N_2)\\ &\leq (N_1 \cap (((Sa) \cap A_1) + A_2)) + (N_2 \cap (((Sa) \cap A_2) + A_1))\\ &\leq ((Sa) \cap A_1) \cap (N_1 + A_2) + ((Sa) \cap A_2) \cap (N_2 + A_1) .\end{aligned}$$

On the other hand

$((Sa) \cap A_1) \cap (N_1 + A_2) = (Sa_1) \cap (N_1 + A_2) \leq N_1 \cap ((Sa_1) + A_2) \leq (Sa_1) \cap (N_1 + A_2)$ implies $(Sa_1) \cap (N_1 + A_2) = N_1 \cap ((Sa_1) + A_2) = (Sa_1) \cap N_1$. Similarly $(Sa_2) \cap (N_2 + A_1) = N_2 \cap ((\mathrm{S}a_2) + A_1) = (Sa_2) \cap N_2$. Since $(Sa_1) \cap N_1$ and $(Sa_2) \cap N_2 \ll N_1$ and N_2 respectively, by Lemma 2.1 (3), $(Sa_1) \cap N_1 + (Sa_2) \cap N_2 \ll N_1 + N_2$. By Lemma 2.1 (2), $(Sa) \cap (N_1 + N_2) \ll N_1 + N_2$. □

Theorem 3.9. Let A be a principally supplemented duo subtractive semimodule. Then every direct summand of A is a principally supplemented semimodule.

Proof. Let $A = A_1 \oplus A_2$ and $a \in A_1$. Then there is K a subsemimodule such that $A = Sa + K$ and $(Sa) \cap K \ll K$. Then $A_1 = Sa + (A_1 \cap K)$. By Lemma 3.6, $K = (K \cap A_1) \oplus (K \cap A_2)$. We prove that $(Sa) \cap (K \cap A_1) \ll K \cap A_1$. Let F be a subsemimodule of $K \cap A_1$ with $K \cap A_1 = (Sa) \cap (K \cap A_1) + F$. Then $K = (Sa) \cap (K \cap A_1) + F + (K \cap A_2) = ((Sa) \cap K) + F + (K \cap A_2)$. Since $(Sa) \cap K \ll K, K = F \oplus (K \cap A_2)$. It follows that $F = K \cap A_1$ this is required. □

Theorem 3.10. Let A be a principally supplemented distributive semimodule. Then every direct summand of A is a principally supplemented semimodule.

Proof. Let $A = A_1 \oplus A_2$ and $a \in A_1$. There is a subsemimodule K of A with $A = Sa + K$ and $(Sa) \cap K \ll K$. Then $A_1 = (Sa) + (A_1 \cap K)$ since A is distributive. By Lemma 2.4, $(Sa) \cap K \ll A_1 \cap K$. □

Definition 3.11 [7]. Let A be a semimodule. Then A is called a principally semisimple if each cyclic subsemimodule is direct summand of A.

Remark 3.12. Each semisimple semimodule is principally semisimple, and each principally semisimple semimodule also principally supplemented.

Proposition 3.13. Let A be a principally supplemented distributive subtractive semimodule. Then $A/Rad(A)$ is a principally semisimple semimodule.

Proof. Since A is subtractive, so $A/Rad(A)$ is a semimodule. Let $a \in A$. There is a subsemimodule A_1 such that $A = Sa + A_1$ and $(Sa) \cap A_1 \ll A_1$. Then $A/Rad(A) = [(Sa + Rad(A))/Rad(A)] + [(A_1 + Rad(A))/Rad(A)]$. Now we prove that $(Sa + Rad(A)) \cap (A_1 + Rad(A)) = Rad(A)$. The distributivity of A shows $\left(Sa + Rad(A)\right) \cap \left(A_1 + Rad(A)\right) = (Sa) \cap A_1 + Rad(A)$. Since $(Sa) \cap A_1 \ll A_1$, so small in A, $(Sa) \cap A_1 \leq Rad(A)$. Thus, $A/Rad(A) = [(Sa + Rad(A))/Rad(A)] \oplus [A_1 + Rad(A)/Rad(A)]$. So any principal subsemimodule of $A/Rad(A)$ is direct summand. □

Theorem 3.14. Let A be a principally supplemented semimodule. Then $= A_1 \oplus A_2$, when A_1 is semisimple semimodule with A_2 is a semimodule and $Rad(A_2) \ll A_2$.

Proof. By Zorn's Lemma we may find a subsemimodule A_1 of A such that $ad(A) \oplus A_1 \ll A$. We prove A_1 is semisimple. Let $a \in A_1$. Since A is principally supplemented, there is a subsemimodule L of A such that $A = Sa + L$ and $(Sa) \cap L \ll L$. Then $(Sa) \cap L = 0$. Let N be a maximal subsemimodule of Sa. If N is unique maximal subsemimodule in Sa, then it is small, hence small in Sa and so in A. This is not possible since $(Sa) \cap Rad(A) = 0$. Hence there is $t \in Sa$ such that $Sa = N + St$. We assume that $N \cap (St) = 0$. Apart from that, let $0 \neq t_1 \in U \cap (St)$. By hypothesis there exists F_1 such that $A = St_1 + F_1$ with $(St_1) \cap F_1$ is small in A. Thus $A = St_1 \oplus F_1$ since $(St_1) \cap F_1 \leq U \cap Rad(A) = 0$. Therefore $Sa = St_1 \oplus ((Sa) \cap F_1)$ and $U = St_1 \oplus (U \cap F_1)$. If $U \cap F_1$ is nonzero, let $0 \neq t_2 \in U \cap F_1$. By hypothesis there exists F_2 such that $A = St_2 + F_2$ with $(St_2) \cap F_2$ is small in A. Thus $A = St_2 \oplus F_2$ since $(St_2) \cap F_2 \leq U \cap Rad(A) = 0$. Then $U \cap F_1 = (St_2) \oplus (U \cap F_1 \cap F_2)$. Thus $Sa = St_1 \oplus St_2 \oplus ((Sa) \cap F_1 \cap F_2)$ and $U = St_1 \oplus St_2 \oplus (U \cap F_1 \cap F_2)$. If $U \cap F_1 \cap F_2$ is nonzero, similarly there exists $0 \neq t_3 \in U \cap F_1 \cap F_2$ and $F_3 \leq A$ such that $A = St_3 \oplus F_3$. Then $Sa = St_1 \oplus St_2 \oplus St_3 \oplus ((Sa) \cap F_1 \cap F_2 \cap F_3)$ and $U = St_1 \oplus St_2 \oplus St_3 \oplus (U \cap F_1 \cap F_2 \cap F_3)$. This process must terminate at a finite step, say n. At this step $Sa = St_1 \oplus St_2 \oplus St_3 \oplus \ldots \oplus St_n$ and so $Sa = U$ since at n^{th} step we must have $U \cap F_1 \cap F_2 \cap ... \cap F_n \leq (Sa) \cap F_1 \cap F_2 \cap ... \cap F_n = 0$. This is contradiction. There is $t \in Sa$ such that $Sa = U \oplus (St)$. Then St is simple semimodule. Therefore each cyclic subsemimodule of A_1 contains a simple subsemimodule. As in the proof of [1, Lemma 9.2], we show that A_1 is semisimple. □

Definition 3.15 [7]. A semimodule A is called principally hollow (P-hollow) if every proper cyclic subsemimodule of A is small in A.

Proposition 3.16. For an indecomposable semimodule A. Consider the next statements:

(1) A is a principally lifting semimodule.

(2) A is a principally hollow semimodule.

(3) A is a principally supplemented semimodule.

Then (1) $\Leftrightarrow$ (2) and (2) $\Longrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (2) Let X be a proper cyclic subsemimodule of A. By (1), there exists a decomposition $A = N \oplus K$ such that $N \leq X$ and $X \cap K \ll K$. Thus $A = X + K$. By assumption, either $K = 0$ or $N = 0$. If $K = 0$, then $A = X + 0 = X$ contradiction. Thus $N = 0$, and then $A = N + K = 0 + K = K$, hence $X = X \cap A = X \cap K \ll K = A$, i.e., $X \ll A$. Therefore (2), holds.

(2) $\Longrightarrow$ (1) Let $a \in A$ and Sa be a proper cyclic subsemimodule of A. By (2) $Sa \ll A$. We will take $U = 0$ and $V = A$ to show that $A = U \oplus V$, $U \leq Sa$ and $(Sa) \cap V \ll V$. Therefore (1) holds.

(2) $\Longrightarrow$ (3) Let $a \in A$. By (2) every cyclic subsemimodule is hollow. Thus $A = Sa + A$ and $(Sa) \cap A \ll A$. So A is a principally supplemented. □

Note that Proposition 3.16, (3) $\Longrightarrow$ (2) does not hold in general. There exists an indecomposable principally supplemented semimodule but not principally hollow.

Example 3.17. Let F be a semifield and x and y commuting indeterminates on F. Suppose a polynomial semiring $S = F[x\ , y]$, the ideals $I_1 =(x^2)$ and $I_2 =(y^2)$ of S, and a semiring $R = S/(x^2, y^2)$. When $A = R\overline{x} + R\overline{y}$. Therefore A is indecomposable R-semimodule, principally supplemented but not principally hollow.

The concept of weakly principally supplemented module introduce in [2]. Similarly, we now give the following definition:

Definition 3.18. Let A be a semimodule. A is called weakly principally supplemented semimodule if for each $a \in A$ there exists a subsemimodule H such that $A = Sa + H$ and $(Sa) \cap H \ll A$.

Remark 3.19. Every weakly supplemented semimodule is weakly principally supplemented.

Definition 3.20 [7]. The semimodule A is called a $\oplus$-principally supplemented if for each $a \in A$ there is a direct summand K of A such that $A = Sa + K$ and $(Sa) \cap K \ll K$.

Remark 3.21. (1) Every $\oplus$-supplemented semimodule is $\oplus$-principally supplemented and obviously every $\oplus$-principally supplemented is weakly principally supplemented. Through a forthcoming work authors looking for the correlations among principally supplemented semimodules, weakly principally supplemented semimodules and $\oplus$-principally supplemented semimodules at finely.

Definition 3.22 [7]. A semimodule A is said to have the summand sum property (SSP) if the sum of two direct summands of A is again a direct summand of A.

Definition 3.23 [7]. A semimodule A is called refinable if for any subsemimodules F, H of A with $A = F\ + H$, there exists a direct summand $F'$of A such that $F' \leq\ F$ and $A = F' + H$.

Theorem 3.24. Consider the following cases, a refinable semimodule A.

(1) A is principally lifting.

(2) A is principally $\oplus$-supplemented.

(3) A is principally supplemented.

(4) A is principally weakly supplemented.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (2).

If A has the summand sum property then (4) $\Longrightarrow$ (1).

Proof. By definitions (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) already clear.

(4) $\Longrightarrow$ (2) Suppose A is principally weakly supplemented semimodule with $a \in A$. By (4) there is a subsemimodule P of A with $A = Sa + P$ and $(Sa) \cap P \ll A$. By hypothesis there exists a direct summand

H of A with $H \leq P$ and $A = Sa + H = H' \oplus H$ for some subsemimodule H' of A. We claim that $(Sa) \cap H \ll H$. For if $(Sa) \cap H + K = H$ for some subsemimodule K of H, then $A = H' + ((Sa) \cap H) + K = H' \oplus K$ as $(Sa) \cap H \ll A$. So $K = H$. Therefore A is principally $\oplus$-supplemented.

(4) $\Longrightarrow$ (1) Suppose A has the summand sum property and let $a \in A$. By (4) there exists a subsemimodule P such that$A = Sa + P$ and $(Sa) \cap P \ll A$. By assumption there is a direct summand $H_1$of A such that H_1 is contained in P and $A = Sa + H_1 = H_1' \oplus H_1$. Since H_1 is direct summand and $(Sa) \cap P \ll A$, $(Sa) \cap H_1 \ll H_1$. Again by assumption there is a direct summand H_2 of A such that H_2 is contained in Sa and A= $H_1 + H_2 = H_2 \oplus H_2'$. By the summand sum property $H_2 \cap H_1$ is a direct summand of A, $A = (H_2 \cap H_1) \oplus K$ for some subsemimodule K of A. Then $H_1 = (H_1 \cap H_2) \oplus (K \cap H_1)$ and $A = H_2 \oplus (K \cap H_1)$. It is evident that $(Sa) \cap (K \cap H_1)$ is small in $K \cap H_1$ since $(Sa) \cap (K \cap H_1) \leq (Sa) \cap H_1 \leq H_1$ and $(Sa) \cap H_1$ is small in H_1, $(Sa) \cap (K \cap H_1) \ll H_1$ and so small in $K \cap H_1$ as $K \cap H_1$ is a direct summand of A. □

4 APPLICATIONS

In this section, we introduce with a study some properties of principally semiperfect semimodules.

Definition 4.1 [13]. A homomorphism $f\ A \longrightarrow B$ of left S-semimodules is called k-quasiregular if whenever $K \leq A$, $a \in A \backslash K$, $a' \in K$, and $f(a) = f(a')$ there exists $s \in Ker\ f$ such that $a = a' + s$.

Definition 4.2 [13]. Let A be a left R-semimodule. A left R-semimodule P together with a R-homomorphism $f: P \longrightarrow A$ is called a projective cover of A if:

(1) P is projective,

(2) f is small, epimorphic and k-quasiregular.

Definition 4.3 [12]. A semiring S is called perfect if every S-semimodule (or every simple S-semimodule) has a projective cover. A semiring is called semiperfect if every finitely generated semimodule has a projective cover.

Definition 4.4 [7]. A semimodule A is called semiperfect if every factor semimodule of A has a projective cover. A semiring S is called semiperfect in case the left S-semimodule S is semiperfect.

Definition 4.5 [7]. A semimodule A is called principally semiperfect if every factor semimodule of A by a cyclic subsemimodule has a projective cover. A semiring S is called principally semiperfect in case the left S-semimodule S is principally semiperfect.

Remark 4.6. Every semiperfect semimodule is principally semiperfect.

Theorem 4.7. Let A be a projective subtractive S-semimodule such that every S-epimorphism of A into a factor semimodule of A is k-quasiregular. Then the following are equivalent:

(1) A is principally semiperfect.

(2) A is principally supplemented.

Proof. (1) $\Longrightarrow$ (2) Let $a \in A$. Since A is a subtractive semimodule, so A/Sa is an S-semimodule [8, p.165]. By (1), A/Sa has a projective cover $h: T \longrightarrow A/Sa$. There exists $g: T \longrightarrow A$ such that $h = \pi g$, where $\pi: A \longrightarrow A/Sa$ is the natural epimorphism. Let $a \in A$. There is $x \in T$ with $\pi(a) = h(x)$ since h is epimorphism. So $\pi(a) = h(x) = \pi(g(x))$. Since π is the natural epimorphism of A then by assumption π is k-quasiregular. Hence, there exists $r \in ker(\pi) = Sa$ such that $a = g(x) + r$. Therefore $A = g(T) + Sa$. We prove $g(T) \cap (Sa)$ is small in $g(T)$. It suffices to show that $g(T) \cap (Sa) = g(ker(h))$ since $ker(h)$ is small in T and any homomorphic image of small semimodules is small under epimorphic maps. Let $x \in ker(h)$. Then $\pi g(x) = h(x) = 0$. therefore $g(x) \in ker(\pi) = Sa$. Hence $g(ker(h)) \leq g(T) \cap (Sa)$. Let $sa \in g(T) \cap (Sa)$ and $g(x) = sa$ for some $x \in T$. Then $h(x) = \pi(g(x)) = \pi(sa) = 0$. Therefore $x \in ker(h)$ and so $g(T) \cap (Sa) \leq g(Ker(h)$. It follows that $g(T) \cap (Sa) = g(ker(h))$ and $g(T)$ is a supplement of Sa.

(2) $\Longrightarrow$ (1) Let $a \in A$. By (2) there is $N \leq A$ such that $A = Sa + N$ and $(Sa) \cap N$ is small in N. Let $f: A \longrightarrow A/(Sa)$ defined by $f(y) = n$ where $y = sa + n$ with $sa \in Sa$, $n \in N$, and $\pi: A \longrightarrow A/(Sa)$ the natural epimorphism. There exists $g: A \longrightarrow A$ with $fg = \pi$. Then $A = g(A) + (Sa) \cap N$. Hence $A = g(A) \cong A/Ker(g)$. As A is projective $A = ker(f) \oplus L$ and L is projective. Let $(fg)_L$ denote the restriction of fg on L. Then $ker(fg)_L = (Sa) \cap N$ and so $(fg)_L: L \longrightarrow A/(Sa)$ is a projective cover of A. □

Similar to the notion of semiregular rings of [2], we now give the definition of semiregular semiring.

Definition 4.8. Let S be a semiring. S is said to be semiregular semiring if every cyclic presented S-semimodule has a projective cover.

Theorem 4.9. Let S be a subtractive semiring such that every S-epimorphism of a S-semimodule S into a factor semimodule of S is k-quasiregular the following are equivalent:

(1) S is principally semiperfect.

(2) S is principally lifting.

(3) S is semiregular.

(4) S is principally supplemented.

Proof. (1) $\Longrightarrow$ (2) Assume that $b \in S$. Since S is a subtractive semiring, so S/Sb is an S-semimodule [8, p.165]. By (1) S/Sb has a projective cover $T \xrightarrow{h} S/Sb$ so that $(h) \ll T$. Let $S \xrightarrow{f} S/Sb$ be the natural epimorphism. Then there exists a map g such that $h = fg$. Then $S = g(T) + Sb$ and $g(T) \cap (Sb) = g\big(ker(h)\big) \ll g(T)$ since homomorphic images of small subsemimodules are small.

(2) $\Longrightarrow$ (3) Suppose S is principally lifting. Let $b \in S$. Then there exists a direct summand left ideal H of Sb with $S = H \oplus K$ and $(Sb) \cap K \ll K$. Then $Sb = H \oplus (Sb) \cap K$ and $(Sb) \cap K \ll A$. By [18, Theorem 3.5], S is semiregular.

(3) $\Longrightarrow$ (4) Suppose S is semiregular. Let $b \in S$ and $f: S \longrightarrow S/Sb$ natural epimorphism. By assumption S/Sb has a projective cover $h: T \longrightarrow S/Sb$. There is $g: T \longrightarrow S$ such that $h = fg$. Then $S = g(T) + (Sb)$ and $g(T) \cap (Sb) \ll g(T)$ since $g(T) \cap (Sb) = g(Ker(h)$ and $Ker(h) \ll T$. Therefore S is principally supplemented.

(4) $\Longrightarrow$ (1) By Theorem 4.7, is clear (Since S is a projective S-semimodule). □

Example 4.10. Let $S = \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a,\ b,\ c \in \mathbb{Z}_4\}$ denote the semiring of upper triangular matrices over integers. It is easy to check that principal left ideals of S as either small in S or a direct summands of S. Therefore S is principally supplemented left S-semimodule. Let e_{12} denote the matrix unit having 1 at (1, 2) and zero elsewhere. Let $I = e_{12}S$. Then I is small left ideal and Jacobson radical $J(S)$ of S is equal to I. Therefore $S/J(S)$ is not semisimple. Hence S is not semiperfect semiring.

Theorem 4.11. Let A be a projective subtractive semimodule with $Rad(A)$ is small in A. Consider following conditions:

(1) A is principally supplemented.

(2) $A/Rad(A)$ is principally semisimple.

Then (1) $\Longrightarrow$ (2). If A is refinable semimodule then (2) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2) Since A is principally supplemented semimodule, $A/Rad(A)$ is principally semisimple by Proposition 3.13.

(2) $\Longrightarrow$ (1) Assume that Sa be any cyclic subsemimodule of A. By (2) There is a subsemimodule K of A such that $A/Rad(A) = [((Sa) + Rad(A))/Rad(A)] \oplus [K/Rad(A)]$. Then $T = (Sa) + K$ and $((Sa) + Rad(T)) \cap K = (Sa) \cap K + Rad(T)] = Rad(T)$. Since $T = (Sa) + K$, being A refinable there is a direct summand N of A such that $N \leq K$ and $A = (Sa) + K = (Sa) + N = L \oplus N$. $(Sa) \cap K \ll A$, so, it is small in K. Thus K is a direct summand. □

5 CONCLUSION

In this paper, we have defined and studied the concepts of principally supplemented (lifting) semimodules. We observed that if A is an indecomposable semimodule, then A is a principally lifting semimodule if and only if A is a principally lifting. Let S be a subtractive semiring such that every S-

epimorphism of a S-semimodule S into a factor semimodule of S is k-quasiregular then S is principally semiperfect if and only if S is principally lifting if and only if S is semiregular if and only if S is principally supplemented.

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