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Unique Colorability of the Distinguishing Coloring

Meysam Korivand[1] and Nasrin Soltankhah

Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran.

Abstract

Let G be a simple graph. A partition $\{V_1, \dots, V_\ell\}$ of vertex set $V(G)$ is distinguishing coloring if it provides a proper coloring and there is no non-trivial automorphism f of G with $f(V_i) = V_i$ for all $i = 1, \dots, \ell$. A graph is called uniquely distinguishing colorable if there is only one partition of vertices of the graph that forms distinguishing coloring with the smallest possible colors. Here, we prove some results on the unique colorability of the distinguishing coloring of a graph, because of their applications in computing the distinguishing chromatic number of disconnected graphs. We introduce two families of uniquely distinguishing colorable graphs, named them type 1 and type 2, and show that every disconnected uniquely distinguishing colorable graphs are the union of two isomorphic graphs of type 2. Also, we present the characterization of all graphs G of order n with property that $\chi_D(G \cup G) = \chi_D(G) = k$, where $k = n-1, n$.

Keywords: uniquely colorable graph, distinguishing colorable graph, automorphism of graph

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1 Introduction and preliminaries

Let G be a simple graph with vertex set $V(G)$. A coloring (or labeling) of a graph G is a partition of the vertices of G into classes, called the color classes. If a coloring contains exactly n disjoint non-empty color classes, then it is called an n-coloring. We say that a coloring with color classes $\{V_1, \dots, V_\ell\}$ of G is distinguishing labeling if there is no non-trivial automorphism f of G with $f(V_i) = V_i$ for all $i = 1, \dots, \ell$. We denote the minimum such ℓ by $D(G)$ and is called distinguish number of G. A distinguishing labeling is distinguishing coloring (or proper distinguishing coloring) if it provides a proper coloring for G. The distinguishing chromatic number of a graph G, denoted by $\chi_D(G)$, is the minimum ℓ such that $\{V_1, \dots, V_\ell\}$ is distinguishing coloring. In 2006, Collins and Trenk proposed this coloring and called *proper distinguishing coloring* (or *distinguishing coloring*) [**?**]. This coloring has attracted the attention of researchers in a short period of time and many articles have been published about it. In [**?**], Harary, Hedetniemi, and Robinson introduced and studied the uniquely colorable graphs. In their work 'coloring' means that 'proper coloring'.

We say that a graph is uniquely distinguishing n-colorable if it has exactly one distinguishing n-coloring. Furthermore, we say a graph is uniquely distinguishing colorable if there is only one partition of its vertex set into the smallest possible number of distinguishing color classes. Actually a uniquely distinguishing

[1]speaker

colorable graph is a uniquely distinguishing $\chi_D(G)$-colorable. The symbols of distinguishing coloring of G will always denote $[\chi_D(G)]$. Any unexplained basic definitions in graph theory comes from [**?**].

Here, we present some results on uniquely distinguishing graphs, because of their applications in computing the distinguishing chromatic number of disconnected graphs. For this propose, we introduce two families of uniquely distinguishing colorable graphs, named them type 1 and type 2, and show that every disconnected uniquely distinguishing colorable graphs are the union of two isomorphic graphs of type 2. Also, we characterize all graphs G of order n with property that $\chi_D(G \cup G) = \chi_D(G) = k$, where $k = n-1, n$.

2 Results

In this section, we present some results on uniquely distinguishing colorable graphs and distinguishing chromatic number of disconnected graphs. Specificaly, we show that there exist some family of disconnected uniquely distinguishing colorable graph.

Remark 2.1. (1) For positive integers n and m, the graphs K_n, $\overline{K_n}$, $K_{n,m}$ and P_{2n} are uniquely distinguishing colorable. But C_n (for $n \geq 5$) and P_{2n+1} (for $n \geq 2$) are not uniquely distinguishing colorable.

(2) Let G be a uniquely colorable graph and $D(G) = 1$. Then G is uniquely distinguishing colorable and $\chi_D(G) = \chi(G)$.

(3) Let G be a connected graph and $\chi_D(G) = 2$. Then G is uniquely distinguishing colorable.

Let G be a uniquely distinguishing colorable graph with unique coloring $[\chi_D(G)] = \{V_1, V_2, \ldots, V_n\}$. Clearly, in a distinguishing n-coloring of G, one can assign n colors to $V(G)$ with $n!$ different ways. We call each of these assigning ways a labelling of the distinguishing color classes. For the sake of convenience, in the following definition, we introduce two types of uniquely distinguishing colorable graphs.

Definition 2.2. Let G be a connected uniquely distinguishing colorable graph. We say that G has type 1 if for any labelling of the distinguishing color classes of components of $G \cup G$ with $\chi_D(G)$ colors, there exists a non-trivial automorphism which embeds one component into the other and preserves all the distinguishing color class labels. Otherwise, we say that G is of type 2.

For instance, the graph in Figure 1(A) is of type 2, and the graph in Figure 1(B) is of type 1.

For an integer n, let $\overset{n}{\cup} G$ denote the disjoint union of n copies of G.

Lemma 2.3. *Let G be a connected graph. For a positive integer n, if $\overset{n}{\cup} G$ is a uniquely distinguishing colorable graph, then G is uniquely distinguishing colorable.*

Lemma 2.4. *Let G be a connected graph. For a positive integer n, if $\overset{n}{\cup} G$ is a uniquely distinguishing colorable, then $\chi_D(\overset{n}{\cup} G) = \chi_D(G)$.*

Lemma 2.5. *Let G be a graph of type 1. Then $\chi_D(G \cup G) = \chi_D(G) + 1$.*

Lemma 2.6. *Let G be a connected graph. For a positive integer n, if $\overset{n}{\cup} G$ is a uniquely distinguishing colorable graph, then $n \leq 2$.*

In the following, we determine the structure of the disconnected uniquely distinguishing colorable graphs.

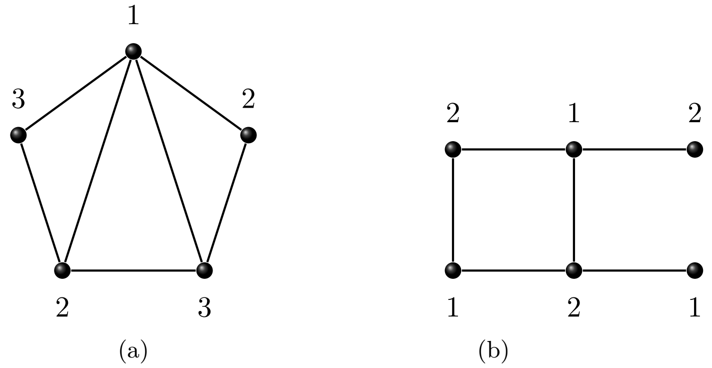

Figure 1: Uniquely distinguishing colorable graphs of of type 2 and type 1.

Theorem 2.7. *Let G be a non-empty uniquely distinguishing colorable graph. Then G is connected or it is isomorphic to $H \cup H$, where H is a connected graph of type* 2.

Proof. Let G be a disconnected uniquely distinguishing colorable graph. By Lemmas **??**, **??**, G has two uniquely distinguishing colorable components, say H_1 and H_2. Consider a distinguishing coloring of G with minimum number of colors. Then H_1 and H_2 have at least one same color. Now suppose that $H_1 \ncong H_2$. Since G has at least one edge, we may assume that $\chi_D(H_1) \geq 2$. Thus H_1 has at least two color classes V_1 and V_2 in $[\chi_D(H_1)]$. Also, let U be a color class of H_2. So we may assume that V_1 and U have a common color. Interchange the colors V_1 and V_2, and produce a new partition of $V(G)$ into distinguishing color classes. This is the required contradiction. Hence we have $H_1 \cong H_2$. Now assume that H_1 (and so H_2) is of type 1. Lemmas **??**, **??** conclude that $H_1 \cup H_2$ is not uniquely distinguishing colorable. This implies that G is a union of copies of a graph of type 2. □

The disconnected graph in Figure 4 is uniquely distinguishing 2-colorable. Clearly, each of its components is of type 2.

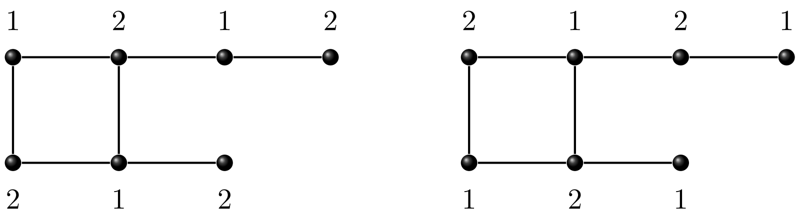

Figure 2: A disconnected uniquely distinguishing colorable graph.

Corollary 2.8. *Let G be a uniquely distinguishing colorable graph. If G is 2-regular, then G is isomorphic to C_3 or C_4.*

Let G be a bipartite graph. If G is an asymmetric graph, i.e., the automorphism group of G is trivial, then G is of type 2, $G \cup G$ is uniquely distinguishing colorable graph and so $\chi_D(G \cup G) = \chi_D(G) = 2$. This means that there are an infinite number of disconnected uniquely distinguishing colorable graphs.

In following, we investigate all graphs G such that $\chi_D(G \cup G) = \chi_D(G)$, when $\chi_D(G) = k$, for $k \in \{|V(G)| - 1, |V(G)|\}$. To do this, we begin with Theorem **??** and Theorem **??** in which all graphs of order n with distinguishing chromatic number n and $n-1$ were charaterized [**?**, **?**].

Theorem 2.9. *[**?**, Theorem 2.3] Let G be a graph. Then $\chi_D(G) = |V(G)|$ if and only of G is a complete multipartite graph.*

Theorem 2.10. *[**?**, Theorem 3.2] Let G be a graph of order $n > 3$. Then $\chi_D(G) = n - 1$ if and only if G is the join of a complete multipartite graph (possibly vacuous) with one of the following:*

(1) $2K_2$, or

(2) $H \cup K_1$, where H is a complete multipartite graph with at least two parts.

In following, we characterise graphs G such that $\chi_D(G \cup G) = \chi_D(G)$, when $\chi_D(G) = k$, for $k = |V(G)| - 1, |V(G)|$.

Theorem 2.11. *Let G be a graph of order n and $\chi_D(G) = n$. Then*

$$\chi_D(G \cup G) = \begin{cases} n+1, & \text{if } G \cong K_n \\ 2n, & \text{if } G \cong \overline{K_n} \\ n, & \text{Otherwise.} \end{cases}$$

Proof. By Theorem **??**, G is a complete multipartite graph. The only non-trivial case is when $G \notin \{K_n, \overline{K_n}\}$. In this case, there is at least one part, via U, with cardinality more than 2. There exist at least two subsets of size $|U|$ of $\{1, 2. \ldots, \chi_D(G)\}$. Color U with $\{1, 2, \ldots, |U|\}$ in a component of $G \cup G$ and color U in the other component with $\{1, 2, \ldots, |U| - 1, |U| + 1\}$. Also, color other vertices of $G \cup G$ arbitrary. One can check that this is a distinguishing $\chi_D(G)$-coloring for $G \cup G$. □

Lemma 2.12. *Let G be the join of a nonvacuous complete multipartite graph with one of graphs (1) and (2) in Theorem **??**. Then $\chi_D(G \cup G) = \chi_D(G)$.*

Theorem 2.13. *Let G be a graph of order $n > 3$ that $\chi_D(G) = n - 1$. Then*

(a) $\chi_D(G \cup G) = \chi_D(G)$ if and only if

(a_1) G is the join of a non-vacuous complete multipartite graph with one of the following:

(a_{11}) $2K_2$, or,

(a_{12}) $H \cup K_1$.

(a_2) $G \cong H \cup K_1$, $H \ncong K_{n-1}$.

(b) $\chi_D(G \cup G) = \chi_D(G) + 1$ if and only if G is one of the following:

(b_1) $2K_2$

(b_2) $K_{n-1} \cup K_1$.

Where H is a complete multipartite graph with at least two parts.

Proof. According to Theorem **??**, assume first that G is the join of a nonvacuous complete multipartite graph with one of the graphs $2K_2$ or $H \cup K_1$. Lemma **??** concludes the results in (a_1). For (b_1), let $G = 2K_2$. Hence, $\chi_D(G \cup G) = \chi_D(4K_2) = 4 = \chi_D(G) + 1$ and the result follows. Let $G = H \cup K_1$. Since H is a complete multipartite graph with at least two parts, $H \ncong \overline{K_{n-1}}$. Now, the results in ($b_2$) and ($a_2$) conclude from Theorem **??**. □

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e-mail: `m.korivand@alzahra.ac.ir or mekorivand@gmail.com`
e-mail: `soltan@alzahra.ac.ir`