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k-Product Cordial Labeling of Splitting Graph of Star Graphs

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Abstract

Let f be a map from $V(G)$ to $\{0,1,...,k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0,1,...,k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x $(x = 0,1,...,k-1)$. A graph that admits k-product cordial labeling is called k-product cordial graph. We have already proved that several families of graphs admit k-product cordial labeling. In this paper, we show that the splitting graph of star graphs admit k-product cordial labeling.

Keywords: Cordial labeling, Product cordial labeling, k-Product cordial labeling, Splitting graph, Star graph

AMS Mathematical Subject Classification [2010]: 05C78

1 Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [3]. The concept of labeling of graph has gained a lot of popularity in the field of graph theory. In 1967, Rosa [14] published a pioneering paper on graph labeling problems.

[1]speaker

Since then, many graph labeling techniques have been introduced and studied by several authors (refer [2]). Cordial labeling is one of the popular labelings defined by Cahit [1] in 1987: Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Followed by this, in 2004 Sundaram et al. [15] introduced a variation of cordial labeling called product cordial labeling: Let f be a function from $V(G)$ to $\{0, 1\}$. For each edge uv, assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i (i = 0, 1)$. Several papers have been published on this topic.

Later in 2012, Ponraj et al. [13] further extended the concept of product cordial labeling and defined k-product cordial labeling as follows: Let f be a map from $V(G)$ to $\{0, 1, ..., k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, ..., k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x $(x = 0, 1, ..., k-1)$. A graph that admits k-product cordial labeling is called k-product cordial graph. We have already proved that several families of graphs admit k-product cordial labeling. The concept of k-product cordial labeling and the work of the authors in [13], motivated us to do further research on this topic. Consequently in our research, we established that the following families of graphs admit k-product cordial labeling: union of graphs [4]; cone and double cone graphs [5]; Napier bridge graphs [6]; product of graphs [7]; powers of paths [8]; fan and double fan graphs [9]; the maximum number of edges in a 4-product cordial graph of order p is $4\lceil\frac{p-1}{4}\rceil\lfloor\frac{p-1}{4}\rfloor + 3$ [10] and path graphs [11]. Jeyanthi et al. [12] already proved that the splitting graph $S'(K_{1,n})$ is a 3-product cordial graph. In this paper, we show that splitting graph of star graphs admit k-product cordial labeling for $k \geq 4$.

We use the following terminology to prove our main results. The splitting graph $S'(G)$ of a graph G is a graph obtained by adding a new vertex v' to each vertex v of G such that v' is adjacent to every vertex that is adjacent to v in G. A bipartite graph is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. If every vertex of V_1 is adjacent with every vertex of V_2, then G is a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. The graph $K_{1,n}$ is called a star graph. An illustration of $S'(K_{1,4})$ is shown in Figure 1.

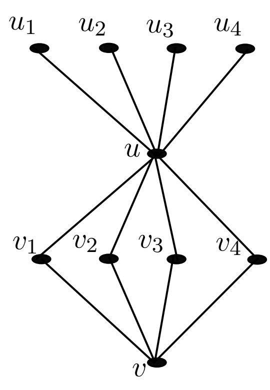

Figure 1: $S'(K_{1,4})$

2 Main Results

Theorem 2.1. *The graph $S'(K_{1,n})$ is k-product cordial for $n \geq \frac{k}{2}$ where k is even and $k \geq 4$.*

Proof. Let the vertex set and the edge set of $S'(K_{1,n})$ be $V(S'(K_{1,n})) = \{u, v, u_i, v_i\ ;\ 1 \leq i \leq n\}$ and $E(S'(K_{1,n})) = \{(u, u_i)\ ;\ 1 \leq i \leq n\}\ \cup\ \{(v, v_i)\ ;\ 1 \leq i \leq n\} \cup\ \{(u, v_i)\ ;\ 1 \leq i \leq n\}$ respectively. We have the following four cases.

Define $f : V(S'(K_{1,n})) \to \{0, 1, 2, ..., k-1\}$ for $k \geq 4$ as follows:

Case (i): If $n \equiv 0 (mod\ k)$, then

$f(u) = 1,\ f(v) = k-1,\ f(u_i) = i(mod\ k)$ and $f(v_i) = i(mod\ k)$ for $1 \leq i \leq n$.

Case (ii): If $n \equiv k-1(mod\ k)$, then

$f(u) = 1,\ f(u_{n-1}) = 0,\ f(u_n) = 2,\ f(v) = k-1,\ f(v_{n-1}) = 0,\ f(v_n) = 1,\ f(u_i) = i(mod\ k)$ and $f(v_i) = i(mod\ k)$ for $1 \leq i \leq \lfloor \frac{n}{k} \rfloor k$.

For $i = \lfloor \frac{n}{k} \rfloor k + j;\ 1 \leq j \leq k-3$,

$f(u_i) = j+1$ and $f(v_i) = j+2$.

Case (iii): If $n \equiv x(mod\ k);\ 1 \leq x \leq \frac{k}{2} - 1$, then

$f(u) = 1,\ f(v) = k-1,\ f(u_i) = i(mod\ k)$ and $f(v_i) = i(mod\ k)$ for $1 \leq i \leq k\left(\lfloor \frac{n}{k} \rfloor - 1\right)$.

For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + j;\ 1 \leq j \leq 2x$,

$$f(u_i) = \begin{cases} \frac{k}{2} & \text{if } j = k-3 \\ 0 & \text{if } j = k-2. \end{cases}$$

$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \ is\ odd \\ k - 1 - \frac{j}{2} & \text{if } j \ is\ even. \end{cases}$$

For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + 2x + j;\ 1 \leq j \leq n - (k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + 2x)$,

$$f(u_i) = \begin{cases} 0 & \text{if } j = k-2 \\ 1 & \text{if } j = k-1. \end{cases}$$

$$f(u_i) = \begin{cases} k - 1 - \frac{j+1}{2} & \text{if } j \ is\ odd \\ 1 + \frac{j}{2} & \text{if } j \ is\ even. \end{cases}$$

For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + j;\ 1 \leq j \leq k-3,\ f(v_i) = j+1$.

For $i = k\lfloor \frac{n}{k} \rfloor - 3 + j;\ 1 \leq j \leq n - (k\lfloor \frac{n}{k} \rfloor - 3)$,

$$f(v_i) = \begin{cases} k-1 & \text{if } j = 1, 4 \\ 1 & \text{if } j = 2, 5 \\ 0 & \text{if } j = 3, 6 \\ \frac{k}{2} & \text{if } j = 7. \end{cases}$$

If $j \geq 8$, then $f(v_i) = \begin{cases} \frac{k}{2} - \frac{j-6}{2} & \text{if } j \ is\ even \\ \frac{k}{2} + \frac{j-7}{2} & \text{if } j \ is\ odd. \end{cases}$

Case (iv): If $n \equiv k - x(mod\ k);\ 2 \leq x \leq \frac{k}{2}$, then

$f(u) = 1,\ f(v) = k-1,\ f(u_i) = i(mod\ k)$ and $f(v_i) = i(mod\ k)$ for $1 \leq i \leq k\lfloor \frac{n}{k} \rfloor$.

For $i = k\lfloor \frac{n}{k} \rfloor + j;\ 1 \leq j \leq k - 2x$,

$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \ is\ odd \\ k - 1 - \frac{j}{2} & \text{if } j \ is\ even. \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + k - 2x + j;\ 1 \leq j \leq x$,

$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \ is\ odd \\ k - 1 - \frac{j}{2} & \text{if } j \ is\ even. \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + j;\ 1 \leq j \leq 4$,

$$f(v_i)=\begin{cases}0 & \text{if } j=1\\ 1 & \text{if } j=2\\ k-1 & \text{if } j=3\\ \frac{k}{2} & \text{if } j=4.\end{cases}$$

For $i=k\lfloor\frac{n}{k}\rfloor+4+j;\ 1\leq j\leq n-(k\lfloor\frac{n}{k}\rfloor+4)$,

$$f(v_i)=\begin{cases}\frac{k}{2}+\frac{j+1}{2} & \text{if } j \textit{ is odd}\\ \frac{k}{2}-\frac{j}{2} & \text{if } j \textit{ is even.}\end{cases}$$

From the above cases, we have

for $n\equiv x(mod\ k);\ x=\frac{k}{2}-1\ or\ k-1,\ v_f(i)=\frac{2n+2}{k}$ for $0\leq i\leq k-1$.

If $n\equiv x(mod\ k);\ 0\leq x\leq\frac{k}{2}-2$, then

$$v_f(i)=\begin{cases}\lfloor\frac{2n+2}{k}\rfloor+1 & \text{if } 1\leq i\leq x+1,\ k-(x+1)\leq i\leq k-1\\ \lfloor\frac{2n+2}{k}\rfloor & \text{if } i=0\ or\ x+2\leq i\leq k-(x+2).\end{cases}$$

If $n\equiv(\frac{k}{2}+x)(mod\ k);\ 0\leq x\leq\frac{k}{2}-2$ and $k>4$, then

$$v_f(i)=\begin{cases}\lfloor\frac{2n+2}{k}\rfloor+1 & \text{if } 1\leq i\leq x+1,\ k-(x+1)\leq i\leq k-1\\ \lfloor\frac{2n+2}{k}\rfloor & \text{if } i=0\ or\ x+2\leq i\leq k-(x+2).\end{cases}$$

If $k=4$ and $n\equiv 2(mod\ 4)$, then

$$v_f(i)=\begin{cases}2\lfloor\frac{n}{k}\rfloor+2 & \text{if } i=1,2\\ 2\lfloor\frac{n}{k}\rfloor+1 & \text{otherwise.}\end{cases}$$

Case 1: If $n\equiv 0(mod\ k)$, then $e_f(i)=\frac{3n}{k}$ for $0\leq i\leq k-1$.

Case 2: If $n\equiv k-1(mod\ k)$, then

$$e_f(i)=\begin{cases}3\lfloor\frac{n}{k}\rfloor+2 & \text{if } i=1,k-1,k-2\\ 3\lfloor\frac{n}{k}\rfloor+3 & \text{otherwise.}\end{cases}$$

Case 3: If $n\equiv x(mod\ k);\ 1\leq x\leq\lfloor\frac{k}{3}\rfloor;\ k>4$, then for $x=1$

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{if } i=1,2,k-2\\ 3(\lfloor\frac{n}{k}\rfloor-1)+3 & \text{otherwise.}\end{cases}$$

for $k=4$ and $x=1$, then

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{if } i=0,1,2\\ 3(\lfloor\frac{n}{k}\rfloor-1)+3 & \text{otherwise.}\end{cases}$$

for $k=6$ and $x=2$,

$e_f(i)=3(\lfloor\frac{n}{k}\rfloor-1)+4;\ 0\leq i\leq 5$,

for $k>6$ and $x=2$,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{if } i=1,2,3,k-1,k-2,k-3\\ 3(\lfloor\frac{n}{k}\rfloor-1)+3 & \text{otherwise.}\end{cases}$$

for $x=3$,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{if } i=0,1,2,3,4,k-1,k-2,k-3,k-4\\ 3(\lfloor\frac{n}{k}\rfloor-1)+3 & \text{otherwise.}\end{cases}$$

for $4\leq x\leq\lfloor\frac{k}{3}\rfloor$ and x is odd,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{if } i=0,1,k-1,2,k-2,...,x+1,k-(x+1),\\ & \frac{k}{2},\frac{k}{2}+1,\frac{k}{2}-1,...,\frac{k}{2}-(\lfloor\frac{x}{2}\rfloor-2),\frac{k}{2}+(\lfloor\frac{x}{2}\rfloor-1)\\ 3(\lfloor\frac{n}{k}\rfloor-1)+3 & \text{otherwise.}\end{cases}$$

for $4\leq x\leq\lfloor\frac{k}{3}\rfloor$ and x is even,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{if } i=0,1,k-1,2,k-2,...,x+1,k-(x+1),\\ & \frac{k}{2},\frac{k}{2}+1,\frac{k}{2}-1,...,\frac{k}{2}+(\frac{x}{2}-2),\frac{k}{2}-(\frac{x}{2}-2)\\ 3(\lfloor\frac{n}{k}\rfloor-1)+3 & \text{otherwise.}\end{cases}$$

Subcase 1: If $n\equiv x(mod\ k)$ where $k\equiv 0(mod\ 3)$, $\lfloor\frac{k}{3}\rfloor+1\leq x\leq\frac{k}{2}-2$ and x is odd,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=\frac{k}{3}+2,k-(\frac{k}{3}+2),\frac{k}{3}+3,k-(\frac{k}{3}+3),...,x+1,k-(x+1),\\ & \frac{k}{2}+(\lfloor\frac{k}{6}\rfloor-1),\frac{k}{2}-(\lfloor\frac{k}{6}\rfloor-1),...,\frac{k}{2}-(\lfloor\frac{x}{2}\rfloor-2),\frac{k}{2}+(\lfloor\frac{x}{2}\rfloor-1)\\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

for $\lfloor\frac{k}{3}\rfloor+1\leq x\leq\frac{k}{2}-1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), ..., x+1, k-(x+1), \\ & \frac{k}{2}+(\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2}-(\lfloor \frac{k}{6} \rfloor - 1), ..., \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 2: If $n \equiv x(mod\ k)$ where $k \equiv 0(mod\ 3)$, $x = \frac{k}{2} - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), ..., x, k-x, x+1, 0, \\ & \frac{k}{2}+(\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2}-(\lfloor \frac{k}{6} \rfloor - 1), ..., \frac{k}{2}-(\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2}+(\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \frac{k}{2} - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), ..., x, k-x, x+1, 0, \\ & \frac{k}{2}+(\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2}-(\lfloor \frac{k}{6} \rfloor - 1), ..., \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 3: If $n \equiv x(mod\ k)$ where $k \equiv 1(mod\ 3)$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = x+1, k-(x+1), \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv x(mod\ k)$ where $k \equiv 1(mod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2} - 2$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor+2, k-(\lfloor \frac{k}{3} \rfloor+2), \lfloor \frac{k}{3} \rfloor+3, k-(\lfloor \frac{k}{3} \rfloor+3), ..., x+1, k-(x+1), \\ & \frac{k}{2}+(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), \frac{k}{2}-(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), ..., \frac{k}{2}-(\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2}+(\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2} - 2$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor+2, k-(\lfloor \frac{k}{3} \rfloor+2), \lfloor \frac{k}{3} \rfloor+3, k-(\lfloor \frac{k}{3} \rfloor+3), ..., x+1, k-(x+1), \\ & \frac{k}{2}+(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), \frac{k}{2}-(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), ..., \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 5: If $n \equiv x(mod\ k)$ where $k \equiv 1(mod\ 3)$, $x = \frac{k}{2} - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor+2, k-(\lfloor \frac{k}{3} \rfloor+2), \lfloor \frac{k}{3} \rfloor+3, k-(\lfloor \frac{k}{3} \rfloor+3), ..., x, k-x, x+1, 0, \\ & \frac{k}{2}+(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), \frac{k}{2}-(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), ..., \frac{k}{2}-(\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2}+(\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \frac{k}{2} - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor+2, k-(\lfloor \frac{k}{3} \rfloor+2), \lfloor \frac{k}{3} \rfloor+3, k-(\lfloor \frac{k}{3} \rfloor+3), ..., x, k-x, x+1, 0, \\ & \frac{k}{2}+(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), \frac{k}{2}-(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), ..., \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 6: If $n \equiv x(mod\ k)$ where $k \equiv 2(mod\ 3)$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = k-(\lfloor \frac{k}{3} \rfloor+2), \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 7: If $n \equiv x(mod\ k)$ where $k \equiv 2(mod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2} - 2$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = k-(\lfloor \frac{k}{3} \rfloor+2), \lfloor \frac{k}{3} \rfloor+3, k-(\lfloor \frac{k}{3} \rfloor+3), ..., x+1, k-(x+1), \\ & \frac{k}{2}-(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor - 1), \frac{k}{2}+(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), ..., \frac{k}{2}-(\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2}+(\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2} - 2$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = k-(\lfloor \frac{k}{3} \rfloor+2), \lfloor \frac{k}{3} \rfloor+3, k-(\lfloor \frac{k}{3} \rfloor+3), ..., x+1, k-(x+1), \\ & \frac{k}{2}-(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor - 1), \frac{k}{2}+(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), ..., \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 8: If $n \equiv x(mod\ k)$ where $k \equiv 2(mod\ 3)$, $x = \frac{k}{2} - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = k-(\lfloor \frac{k}{3} \rfloor+2), \lfloor \frac{k}{3} \rfloor+3, k-(\lfloor \frac{k}{3} \rfloor+3), ..., x, k-x, x+1, 0, \\ & \frac{k}{2}-(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor - 1), \frac{k}{2}+(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), ..., \frac{k}{2}-(\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2}+(\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \frac{k}{2} - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = k-(\lfloor \frac{k}{3} \rfloor+2), \lfloor \frac{k}{3} \rfloor+3, k-(\lfloor \frac{k}{3} \rfloor+3), ..., x, k-x, x+1, 0, \\ & \frac{k}{2}-(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor - 1), \frac{k}{2}+(\lfloor \frac{1}{2}\lfloor \frac{k}{3} \rfloor \rfloor), ..., \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 9: If $k = 8$ and $n \equiv 3(mod\ 8)$, then

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = 0 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Case 4: If $n \equiv k - x(mod\ k)$; $2 \leq x \leq \lfloor \frac{k}{3} \rfloor$ and x is odd, then

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{if } i = \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} - 1, ..., \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & 0, k-1, 1, k-2, 2, ..., (\lfloor \frac{x}{2} \rfloor + 1), k - (\lfloor \frac{x}{2} \rfloor + 2) \\ 3(\lfloor \frac{n}{k} \rfloor) + 3 & \text{otherwise.} \end{cases}$$

for $2 \leq x \leq \lfloor \frac{k}{3} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{if } i = \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} - 1, ..., \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & 0, k-1, 1, k-2, 2, ..., k - (\frac{x}{2} + 1), \frac{x}{2} + 1 \\ 3(\lfloor \frac{n}{k} \rfloor) + 3 & \text{otherwise.} \end{cases}$$

Subcase 1: If $n \equiv k - x(mod\ k)$ where $k \equiv 0(mod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \frac{k}{2}$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\frac{k}{3} - 1), \frac{k}{2} - (\frac{k}{3} - 1), \frac{k}{2} + \frac{k}{3}, \frac{k}{2} - \frac{k}{3}, ..., \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & k - (\lfloor \frac{k}{6} \rfloor + 2), (\lfloor \frac{k}{6} \rfloor + 2), ..., (\lfloor \frac{x}{2} \rfloor + 1), k - (\lfloor \frac{x}{2} \rfloor + 2) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \frac{k}{2}$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\frac{k}{3} - 1), \frac{k}{2} - (\frac{k}{3} - 1), \frac{k}{2} + \frac{k}{3}, \frac{k}{2} - \frac{k}{3}, ..., \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & k - (\lfloor \frac{k}{6} \rfloor + 2), (\lfloor \frac{k}{6} \rfloor + 2), ..., k - (\frac{x}{2} + 1), (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 2: If $n \equiv k - x(mod\ k)$ where $k \equiv 1(mod\ 3)$ and $k > 4$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 3: If $k = 4$, $n \equiv 2(mod\ 4)$, then

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = 1, 3 \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv k - x(mod\ k)$ where $k \equiv 1(mod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2}$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, \frac{k}{2} - \lfloor \frac{k}{3} \rfloor, ..., \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), ..., (\lfloor \frac{x}{2} \rfloor + 1), k - (\lfloor \frac{x}{2} \rfloor + 2) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2}$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor), ..., \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), ..., k - (\frac{x}{2} + 1), (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 5: If $n \equiv k - x(mod\ k)$ where $k \equiv 2(mod\ 3)$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 6: If $n \equiv k - x(mod\ k)$ where $k \equiv 2(mod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2}$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, ..., \frac{k}{2} - (x-2), \frac{k}{2} + (x-2), \\ & (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), \\ & ..., (\lfloor \frac{x}{2} \rfloor + 1), k - (\lfloor \frac{x}{2} \rfloor + 2) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2}$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, ..., \frac{k}{2} - (x-2), \frac{k}{2} + (x-2), \\ & (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), ..., k - (\frac{x}{2} + 1), (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Therefore, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq k-1$. Hence, the graph $S'(K_{1,n})$ is k-product cordial for $n \geq \frac{k}{2}$ where k is even and $k \geq 4$. □

Example 2.2. An example of 8-product cordial labeling of $S'(K_{1,10})$ shown in Figure 2.

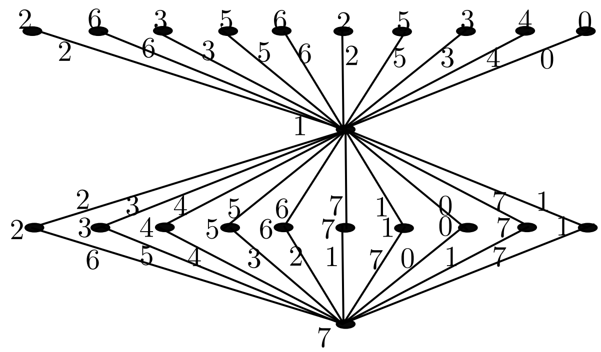

Figure 2: 8-Product cordial labeling of $S'(K_{1,10})$.

Theorem 2.3. *The graph $S'(K_{1,n})$ is k-product cordial for $n \geq \lfloor \frac{k}{2} \rfloor$ where k is odd and $k \geq 4$.*

Proof. Let the vertex set and the edge set of $S'(K_{1,n})$ be $V(S'(K_{1,n})) = \{u, v, u_i, v_i \ ; \ 1 \leq i \leq n\}$ and $E(S'(K_{1,n})) = \{(u, u_i) \ ; \ 1 \leq i \leq n\} \ \cup \ \{(v, v_i) \ ; \ 1 \leq i \leq n\} \cup \ \{(u, v_i) \ ; \ 1 \leq i \leq n\}$ respectively. We have the following four cases.

Define $f : V(S'(K_{1,n})) \rightarrow \{0, 1, 2, ..., k-1\}$ for $k \geq 4$ as follows:

Case (i): If $n \equiv 0(mod\ k)$, then

$f(u) = 1,\ f(v) = k-1,\ f(u_i) = i(mod\ k)$ and $f(v_i) = i(mod\ k)$ for $1 \leq i \leq n$.

Case (ii): If $n \equiv k-1(mod\ k)$, then

$f(u) = 1,\ f(u_{n-1}) = 0,\ f(u_n) = 0,\ f(v) = k-1$.

$f(u_i) = i(mod\ k)$ and $f(v_i) = i(mod\ k)$ for $1 \leq i \leq \lfloor \frac{n}{k} \rfloor k$.

For $i = \lfloor \frac{n}{k} \rfloor k + j;\ 1 \leq j \leq k-3,\ f(u_i) = j+1$.

For $i = \lfloor \frac{n}{k} \rfloor k + j;\ 1 \leq j \leq k-1,\ f(v_i) = j$.

Case (iii): If $n \equiv x(mod\ k);\ 1 \leq x \leq \lfloor \frac{k}{2} \rfloor$, then

$f(u) = 1,\ f(v) = k-1,\ f(u_i) = i(mod\ k)$ and $f(v_i) = i(mod\ k)$ for $1 \leq i \leq k\left(\lfloor \frac{n}{k} \rfloor - 1\right)$.

For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + j;\ 1 \leq j \leq 2x$,

$$f(u_i) = \begin{cases} 0 & \text{if } j = 2\lfloor \frac{k}{2} \rfloor - 1 \\ 1 & \text{if } j = 2\lfloor \frac{k}{2} \rfloor. \end{cases}$$

$$f(u_i) = \begin{cases} k-1-\frac{j+1}{2} & \text{if } j \textit{ is odd} \\ 1+\frac{j}{2} & \text{if } j \textit{ is even.} \end{cases}$$

For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + 2x + j;\ 1 \leq j \leq n - \left(k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + 2x\right)$,

$$f(u_i) = \begin{cases} 0 & \text{if } j = k-2 \\ 1 & \text{if } j = k-1. \end{cases}$$

$$f(u_i) = \begin{cases} k-1-\frac{j+1}{2} & \text{if } j \textit{ is odd} \\ 1+\frac{j}{2} & \text{if } j \textit{ is even.} \end{cases}$$

For $i = k\left(\lfloor \frac{n}{k} \rfloor - 1\right) + j;\ 1 \leq j \leq k-3,\ f(v_i) = 1+j$.

For $i = k\lfloor \frac{n}{k} \rfloor - 3 + j;\ 1 \leq j \leq n - (k\lfloor \frac{n}{k} \rfloor - 3)$,

$$f(v_i) = \begin{cases} k-1 & \text{if } j = 1,4 \\ 1 & \text{if } j = 2,5 \\ 0 & \text{if } j = 3,6. \end{cases}$$

If $j \geq 7$, then $f(v_i) = \begin{cases} \lfloor \frac{k}{2} \rfloor + 1 - \frac{j-5}{2} & \text{if } j \textit{ is odd} \\ \lfloor \frac{k}{2} \rfloor + \frac{j-6}{2} & \text{if } j \textit{ is even.} \end{cases}$

Case (iv): If $n \equiv k-x(mod\ k);\ 2 \leq x \leq \lfloor \frac{k}{2} \rfloor$, then

$f(u) = 1,\ f(v) = k-1,\ f(u_i) = i(mod\ k)$ and $f(v_i) = i(mod\ k)$ for $1 \leq i \leq k\lfloor \frac{n}{k} \rfloor$.

For $i = k\lfloor \frac{n}{k} \rfloor + j$; $1 \leq j \leq 2$, $f(u_i) = 0$.

For $i = k\lfloor \frac{n}{k} \rfloor + 2 + j$; $1 \leq j \leq 2(\lfloor \frac{k}{2} \rfloor - x)$,

$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \textit{ is odd} \\ k - 1 - \frac{j}{2} & \text{if } j \textit{ is even.} \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + 2 + 2(\lfloor \frac{k}{2} \rfloor - x) + j$; $1 \leq j \leq n - (k\lfloor \frac{n}{k} \rfloor + 2 + 2(\lfloor \frac{k}{2} \rfloor - x))$,

$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \textit{ is odd} \\ k - 1 - \frac{j}{2} & \text{if } j \textit{ is even.} \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + j$; $1 \leq j \leq 2$, $f(v_i) = \begin{cases} 1 & \text{if } j = 1 \\ k - 1 & \text{if } j = 2. \end{cases}$.

For $i = k\lfloor \frac{n}{k} \rfloor + 2 + j$; $1 \leq j \leq n - (k\lfloor \frac{n}{k} \rfloor + 2)$,

$$f(v_i) = \begin{cases} \lfloor \frac{k}{2} \rfloor + \frac{j+1}{2} & \text{if } j \textit{ is odd} \\ \lfloor \frac{k}{2} \rfloor + 1 - \frac{j}{2} & \text{if } j \textit{ is even.} \end{cases}$$

From the above cases, we have

for $n \equiv \lfloor \frac{k}{2} \rfloor (mod\ k)$,

$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } i = 1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } i = 0 \textit{ or } 2 \leq i \leq k-1. \end{cases}$$

If $n \equiv x(mod\ k)$; $0 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$, then

$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } 1 \leq i \leq x+1,\ k-(x+1) \leq i \leq k-1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } i = 0 \textit{ or } x+2 \leq i \leq k-(x+2). \end{cases}$$

If $n \equiv k - 1(mod\ k)$, then $v_f(i) = \frac{2n+2}{k}$ for $0 \leq i \leq k-1$.

If $n \equiv (\lfloor \frac{k}{2} \rfloor + x)(mod\ k)$; $1 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$, then

$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } 0 \leq i \leq x,\ k-x \leq i \leq k-1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } x+1 \leq i \leq k-(x+1). \end{cases}$$

Case 1: If $n \equiv 0(mod\ k)$, then $e_f(i) = \frac{3n}{k}$ for $0 \leq i \leq k-1$.

Case 2: If $n \equiv k - 1(mod\ k)$, then

$$e_f(i) = \begin{cases} 3\lfloor \frac{n}{k} \rfloor + 2 & \text{if } i = 0, 1, k-1 \\ 3\lfloor \frac{n}{k} \rfloor + 3 & \text{otherwise.} \end{cases}$$

Case 3: If $n \equiv x(mod\ k)$; $1 \leq x \leq \lfloor \frac{k}{3} \rfloor$, then for $x = 1$

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 1, 2, k-2 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $x = 2$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 1, 2, 3, k-1, k-2, k-3 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $x = 3$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, 2, 3, 4, k-1, k-2, k-3, k-4 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $4 \leq x \leq \lfloor \frac{k}{3} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, k-1, 2, k-2, ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil - 1, \lceil \frac{k}{2} \rceil + 1, ..., \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $4 \leq x \leq \lfloor \frac{k}{3} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, k-1, 2, k-2, ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil - 1, \lceil \frac{k}{2} \rceil + 1, ..., \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

Subcase 1: If $n \equiv x(mod\ k)$ where $k \equiv 0(mod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3} + 2, k - (\frac{k}{3} + 2), \frac{k}{3} + 3, k - (\frac{k}{3} + 3), ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{k}{6} \rfloor - 1), \lceil \frac{k}{2} \rceil - \lfloor \frac{k}{6} \rfloor, \lceil \frac{k}{2} \rceil + \lfloor \frac{k}{6} \rfloor, \\ & ..., \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{k}{6} \rfloor - 1), \lceil \frac{k}{2} \rceil - \lfloor \frac{k}{6} \rfloor, \lceil \frac{k}{2} \rceil + \lfloor \frac{k}{6} \rfloor, ..., \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 2: If $n \equiv x(mod\ k)$ where $k \equiv 0(mod\ 3)$, $x = \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), ..., x, k-x, 0, 1, \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{k}{6} \rfloor - 1), \lceil \frac{k}{2} \rceil - \lfloor \frac{k}{6} \rfloor, \lceil \frac{k}{2} \rceil + \lfloor \frac{k}{6} \rfloor, \\ & ..., \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \lfloor \frac{k}{2} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), ..., x, k-x, 0, 1, \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{k}{6} \rfloor - 1), \lceil \frac{k}{2} \rceil - \lfloor \frac{k}{6} \rfloor, \lceil \frac{k}{2} \rceil + \lfloor \frac{k}{6} \rfloor, ..., \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 3: If $n \equiv x(mod\ k)$ where $k \equiv 1(mod\ 3)$ and $k > 7$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv 3(mod\ k)$ and $k = 7$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = 0, 1 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 5: If $n \equiv x(mod\ k)$ where $k \equiv 1(mod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 6: If $n \equiv x(mod\ k)$ where $k \equiv 1(mod\ 3)$, $x = \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), ..., x, k-x, 0, 1, \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \lfloor \frac{k}{2} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), ..., x, k-x, 0, 1, \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 7: If $n \equiv x(mod\ k)$ where $k \equiv 2(mod\ 3)$ and $k > 5$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 8: If $n \equiv 2(mod\ k)$ where $k = 5$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = 1, \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 9: If $n \equiv x(mod\ k)$ where $k \equiv 2(mod\ 3)$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \lfloor \frac{k}{3} \rfloor + 4, k - (\lfloor \frac{k}{3} \rfloor + 4), ..., x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \lceil \frac{k}{2} \rceil - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \\ & ..., \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor\frac{k}{3}\rfloor+2 \leq x \leq \lfloor\frac{k}{2}\rfloor-1$ and x is even,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=\lfloor\frac{k}{3}\rfloor+3, k-(\lfloor\frac{k}{3}\rfloor+3), \lfloor\frac{k}{3}\rfloor+4, k-(\lfloor\frac{k}{3}\rfloor+4), ..., x+1, k-(x+1), \\ & \lceil\frac{k}{2}\rceil+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor-1), \lceil\frac{k}{2}\rceil-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), ..., \lceil\frac{k}{2}\rceil-(\frac{x}{2}-2), \lceil\frac{k}{2}\rceil+(\frac{x}{2}-2) \\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

Subcase 10: If $n \equiv x(mod\ k)$ where $k \equiv 2(mod\ 3)$, $x=\lfloor\frac{k}{2}\rfloor$ and x is odd,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=\lfloor\frac{k}{3}\rfloor+3, k-(\lfloor\frac{k}{3}\rfloor+3), \lfloor\frac{k}{3}\rfloor+4, k-(\lfloor\frac{k}{3}\rfloor+4), ..., x, k-x, 0, 1, \\ & \lceil\frac{k}{2}\rceil+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor-1), \lceil\frac{k}{2}\rceil-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), \lceil\frac{k}{2}\rceil+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), \\ & ..., \lceil\frac{k}{2}\rceil+(\lfloor\frac{x}{2}\rfloor-2), \lceil\frac{k}{2}\rceil-(\lfloor\frac{x}{2}\rfloor-1) \\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

for $x=\lfloor\frac{k}{2}\rfloor-1$ and x is even,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor-1)+5 & \text{if } i=\lfloor\frac{k}{3}\rfloor+3, k-(\lfloor\frac{k}{3}\rfloor+3), \lfloor\frac{k}{3}\rfloor+4, k-(\lfloor\frac{k}{3}\rfloor+4), ..., x, k-x, 0, 1, \\ & \lceil\frac{k}{2}\rceil+(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor-1), \lceil\frac{k}{2}\rceil-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor), ..., \lceil\frac{k}{2}\rceil-(\frac{x}{2}-2), \lceil\frac{k}{2}\rceil+(\frac{x}{2}-2) \\ 3(\lfloor\frac{n}{k}\rfloor-1)+4 & \text{otherwise.}\end{cases}$$

Case 4: If $n \equiv k-x(mod\ k)$; $2 \leq x \leq \lfloor\frac{k}{3}\rfloor$ and x is odd, then

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{if } i=\lfloor\frac{k}{2}\rfloor, \lfloor\frac{k}{2}\rfloor+1, \lfloor\frac{k}{2}\rfloor-1, ..., \lfloor\frac{k}{2}\rfloor-(x-2), \lfloor\frac{k}{2}\rfloor+(x-1), \\ & 0, k-1, 1, k-2, 2, ..., k-(\lfloor\frac{x}{2}\rfloor+1), (\lfloor\frac{x}{2}\rfloor+1) \\ 3(\lfloor\frac{n}{k}\rfloor)+3 & \text{otherwise.}\end{cases}$$

for $2 \leq x \leq \lfloor\frac{k}{3}\rfloor$ and x is even,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{if } i=\lfloor\frac{k}{2}\rfloor, \lfloor\frac{k}{2}\rfloor+1, \lfloor\frac{k}{2}\rfloor-1, ..., \lfloor\frac{k}{2}\rfloor-(x-2), \lfloor\frac{k}{2}\rfloor+(x-1), \\ & 0, k-1, 1, k-2, 2, ..., \frac{x}{2}, k-(\frac{x}{2}+1) \\ 3(\lfloor\frac{n}{k}\rfloor)+3 & \text{otherwise.}\end{cases}$$

Subcase 1: If $n \equiv k-x(mod\ k)$ where $k \equiv 0(mod\ 3)$, $\lfloor\frac{k}{3}\rfloor+1 \leq x \leq \lfloor\frac{k}{2}\rfloor$ and x is odd,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=\lfloor\frac{k}{2}\rfloor-(\frac{k}{3}-1), \lfloor\frac{k}{2}\rfloor+\frac{k}{3}, \lfloor\frac{k}{2}\rfloor-\frac{k}{3}, ..., \lfloor\frac{k}{2}\rfloor-(x-2), \lfloor\frac{k}{2}\rfloor+(x-1), \\ & k-(\lfloor\frac{k}{6}\rfloor+2), (\lfloor\frac{k}{6}\rfloor+2), ..., k-(\lfloor\frac{x}{2}\rfloor+1), (\lfloor\frac{x}{2}\rfloor+1), \\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

for $\lfloor\frac{k}{3}\rfloor+1 \leq x \leq \lfloor\frac{k}{2}\rfloor$ and x is even,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=\lfloor\frac{k}{2}\rfloor-(\frac{k}{3}-1), \lfloor\frac{k}{2}\rfloor+\frac{k}{3}, \lfloor\frac{k}{2}\rfloor-\frac{k}{3}, ..., \lfloor\frac{k}{2}\rfloor-(x-2), \lfloor\frac{k}{2}\rfloor+(x-1), \\ & k-(\lfloor\frac{k}{6}\rfloor+2), (\lfloor\frac{k}{6}\rfloor+2), ..., \frac{x}{2}, k-(\frac{x}{2}+1) \\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

Subcase 2: If $n \equiv k-x(mod\ k)$ where $k \equiv 1(mod\ 3)$, $x=\lfloor\frac{k}{3}\rfloor+1$,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=\lfloor\frac{k}{2}\rfloor-(\lfloor\frac{k}{3}\rfloor-1), \lfloor\frac{k}{2}\rfloor+(\lfloor\frac{k}{3}\rfloor) \\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

Subcase 3: If $n \equiv k-x(mod\ k)$ where $k \equiv 1(mod\ 3)$, $\lfloor\frac{k}{3}\rfloor+2 \leq x \leq \lfloor\frac{k}{2}\rfloor$ and x is odd,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=\lfloor\frac{k}{2}\rfloor-(\lfloor\frac{k}{3}\rfloor-1), \lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{3}\rfloor, \lfloor\frac{k}{2}\rfloor-\lfloor\frac{k}{3}\rfloor, ..., \lfloor\frac{k}{2}\rfloor-(x-2), \lfloor\frac{k}{2}\rfloor+(x-1), \\ & k-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor+2), (\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor+2), ..., k-(\lfloor\frac{x}{2}\rfloor+1), (\lfloor\frac{x}{2}\rfloor+1) \\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

for $\lfloor\frac{k}{3}\rfloor+2 \leq x \leq \lfloor\frac{k}{2}\rfloor$ and x is even,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=\lfloor\frac{k}{2}\rfloor-(\lfloor\frac{k}{3}\rfloor-1), \lfloor\frac{k}{2}\rfloor+\lfloor\frac{k}{3}\rfloor, \lfloor\frac{k}{2}\rfloor-\lfloor\frac{k}{3}\rfloor, ..., \lfloor\frac{k}{2}\rfloor-(x-2), \lfloor\frac{k}{2}\rfloor+(x-1), \\ & k-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor+2), (\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor+2), ..., \frac{x}{2}, k-(\frac{x}{2}+1) \\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

Subcase 4: If $n \equiv k-x(mod\ k)$ where $k \equiv 2(mod\ 3)$ and $k>5$, $x=\lfloor\frac{k}{3}\rfloor+1$,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=k-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor+2), \\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

Subcase 5: If $n \equiv k-x(mod\ k)$ where $k=5$, $x=2$,

$$e_f(i)=\begin{cases} 3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=2,3 \\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

Subcase 6: If $n \equiv k-x(mod\ k)$ where $k \equiv 2(mod\ 3)$, $\lfloor\frac{k}{3}\rfloor+2 \leq x \leq \lfloor\frac{k}{2}\rfloor$ and x is odd,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=\lfloor\frac{k}{2}\rfloor-\lfloor\frac{k}{3}\rfloor,\lfloor\frac{k}{2}\rfloor+(\lfloor\frac{k}{3}\rfloor+1),\lfloor\frac{k}{2}\rfloor-(\lfloor\frac{k}{3}\rfloor+1),\\ & \quad ...,\lfloor\frac{k}{2}\rfloor-(x-2),\lfloor\frac{k}{2}\rfloor+(x-1),\\ & \quad k-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor+2),\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor+2,...,k-(\lfloor\frac{x}{2}\rfloor+1),(\lfloor\frac{x}{2}\rfloor+1)\\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

for $\lfloor\frac{k}{3}\rfloor+2\le x\le\lfloor\frac{k}{2}\rfloor$ and x is even,

$$e_f(i)=\begin{cases}3(\lfloor\frac{n}{k}\rfloor)+1 & \text{if } i=\lfloor\frac{k}{2}\rfloor-\lfloor\frac{k}{3}\rfloor,\lfloor\frac{k}{2}\rfloor+(\lfloor\frac{k}{3}\rfloor+1),\lfloor\frac{k}{2}\rfloor-(\lfloor\frac{k}{3}\rfloor+1),...,\lfloor\frac{k}{2}\rfloor-(x-2),\lfloor\frac{k}{2}\rfloor+(x-1),\\ & \quad k-(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor+2),(\lfloor\frac{1}{2}\lfloor\frac{k}{3}\rfloor\rfloor+2),...,\frac{x}{2},k-(\frac{x}{2}+1)\\ 3(\lfloor\frac{n}{k}\rfloor)+2 & \text{otherwise.}\end{cases}$$

Therefore, we have $|v_f(i)-v_f(j)|\le 1$ and $|e_f(i)-e_f(j)|\le 1$ for all $0\le i,j\le k-1$. Hence, the graph $S'(K_{1,n})$ is k-product cordial for $n\ge\lfloor\frac{k}{2}\rfloor$ where k is odd and $k\ge 4$. □

Example 2.4. An example of 15-product cordial labeling of $S'(K_{1,11})$ is shown in Figure 3.

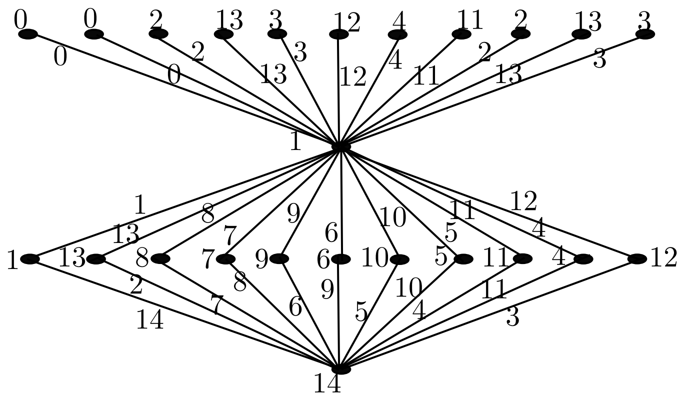

Figure 3: 15-Product cordial labeling of $S'(K_{1,11})$.

Theorem 2.5. *The graph $S'(K_{1,n})$ is k-product cordial if and only if $n\ge 1$ and $k\ge 4$ except $\lfloor\frac{k}{3}\rfloor\le n\le\lfloor\frac{k}{2}\rfloor-1$ if $k\equiv 1,0\ (mod\ 3)$ for $k=4,6$ and $k\ge 8$ and $\lceil\frac{k}{3}\rceil\le n\le\lfloor\frac{k}{2}\rfloor-1$ if $k\equiv 2\ (mod\ 3)$ for $k\ge 8$.*

Proof. From Theorems 2.1 and 2.3, the graph $S'(K_{1,n})$ is k-product cordial if $n\ge\lfloor\frac{k}{2}\rfloor$ and $k\ge 4$.
For $k\ge 5$, $1\le n\le\lfloor\frac{k}{3}\rfloor-1$ if $k\equiv 1,0\ (mod\ 3)$ and $1\le n\le\lceil\frac{k}{3}\rceil-1$ if $k\equiv 2\ (mod\ 3)$.
For k is even and $1\le i\le n$, we assign $f(u)=1$, $f(v)=k-1$, $f(u_1)=0$, $f(u_2)=\frac{k}{2}$ and $f(v_i)=\frac{k}{2}+i$.
For $3\le i\le n$, we assign $f(u_i)=\begin{cases}\frac{i-1}{2}+1 & \text{if } i \textit{ is odd}\\ k-\frac{i}{2} & \text{if } i \textit{ is even.}\end{cases}$
For k is odd and $1\le i\le n$, we assign
$f(u)=1$, $f(v)=k-1$, $f(u_1)=0$, and $f(v_i)=\lfloor\frac{k}{2}\rfloor+i$.
For $2\le i\le n$, we assign $f(u_i)=\begin{cases}\frac{i}{2}+1 & \text{if } i \textit{ is even}\\ k-1-\frac{i-1}{2} & \text{if } i \textit{ is odd.}\end{cases}$
From the above labeling pattern, we have
for $n=1$ and k is even,

$$v_f(i)=\begin{cases}1 & \text{if } i=1,k-1,0,\frac{k}{2}+1\\ 0 & \text{otherwise}\end{cases}$$

for $n=2$ and k is even,

$$v_f(i)=\begin{cases}1 & \text{if } i=1,k-1,0,\frac{k}{2},\frac{k}{2}+1,\frac{k}{2}+2\\ 0 & \text{otherwise.}\end{cases}$$

for $n\ge 3$, n and k are even,

$$v_f(i)=\begin{cases}1 & \text{if } i=0,\frac{k}{2},1,k-1,2,k-2,...,\frac{n}{2},k-\frac{n}{2},\\ & \quad \frac{k}{2}+1,\frac{k}{2}+2,...,\frac{k}{2}+n\\ 0 & \text{otherwise.}\end{cases}$$

for $n \geq 3$, n is odd and k is even,
$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 1, k-1, 2, k-2, ..., k - \lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor + 1, \\ & \quad \frac{k}{2}+1, \frac{k}{2}+2, ..., \frac{k}{2}+n \\ 0 & \text{otherwise.} \end{cases}$$
For $n = 1$ and k is even,
$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}+1, \frac{k}{2}-1 \\ 0 & \text{otherwise} \end{cases}$$
for $n = 2$ and k is even,
$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, \frac{k}{2}+1, \frac{k}{2}-1, \frac{k}{2}+2, \frac{k}{2}-2 \\ 0 & \text{otherwise.} \end{cases}$$
for $n \geq 3$, n and k are even,
$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 2, k-2, ..., \frac{n}{2}, k - \frac{n}{2}, \\ & \quad \frac{k}{2}+1, \frac{k}{2}-1, \frac{k}{2}+2, \frac{k}{2}-2, ..., \frac{k}{2}+n, \frac{k}{2}-n \\ 0 & \text{otherwise.} \end{cases}$$
for $n \geq 3$, n is odd and k is even,
$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 2, k-2, ..., k - \lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor + 1, \\ & \quad \frac{k}{2}+1, \frac{k}{2}-1, \frac{k}{2}+2, \frac{k}{2}-2, ..., \frac{k}{2}+n, \frac{k}{2}-n \\ 0 & \text{otherwise.} \end{cases}$$
For $n = 1$ and k is odd,
$$v_f(i) = \begin{cases} 1 & \text{if } i = 1, k-1, 0, \lfloor\frac{k}{2}\rfloor + 1 \\ 0 & \text{otherwise} \end{cases}$$
for $n \geq 2$, n and k are odd,
$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, 1, k-1, 2, k-2, ..., \lfloor\frac{n}{2}\rfloor + 1, k - (\lfloor\frac{n}{2}\rfloor + 1), \\ & \quad \lfloor\frac{k}{2}\rfloor + 1, \lfloor\frac{k}{2}\rfloor + 2, ..., \lfloor\frac{k}{2}\rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$
for $n \geq 2$, n is even and k is odd,
$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, 1, k-1, 2, k-2, ..., k - \frac{n}{2}, \frac{n}{2} + 1, \\ & \quad \lfloor\frac{k}{2}\rfloor + 1, \lfloor\frac{k}{2}\rfloor + 2, ..., \lfloor\frac{k}{2}\rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$
For $n = 1$ and k is odd,
$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \lfloor\frac{k}{2}\rfloor, \lfloor\frac{k}{2}\rfloor + 1 \\ 0 & \text{otherwise} \end{cases}$$
for $n \geq 2$, n and k are odd,
$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, 2, k-2, 3, k-3, ..., \lfloor\frac{n}{2}\rfloor + 1, k - (\lfloor\frac{n}{2}\rfloor + 1), \\ & \quad \lfloor\frac{k}{2}\rfloor, \lfloor\frac{k}{2}\rfloor + 1, \lfloor\frac{k}{2}\rfloor - 1, \lfloor\frac{k}{2}\rfloor + 2, ..., \lfloor\frac{k}{2}\rfloor - (n-1), \lfloor\frac{k}{2}\rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$
for $n \geq 2$, n is even and k is odd,
$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, 2, k-2, 3, k-3, ..., k - \frac{n}{2}, \frac{n}{2} + 1, \\ & \quad \lfloor\frac{k}{2}\rfloor, \lfloor\frac{k}{2}\rfloor + 1, \lfloor\frac{k}{2}\rfloor - 1, \lfloor\frac{k}{2}\rfloor + 2, ..., \lfloor\frac{k}{2}\rfloor - (n-1), \lfloor\frac{k}{2}\rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$
Therefore, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq k-1$. Hence, the graph $S'(K_{1,n})$ is k-product cordial if $n \geq \lfloor\frac{k}{2}\rfloor$, $1 \leq n \leq \lfloor\frac{k}{3}\rfloor - 1$ for $k \equiv 1, 0\ (mod\ 3)$ and $1 \leq n \leq \lceil\frac{k}{3}\rceil - 1$ for $k \equiv 2\ (mod\ 3)$.

Now, for $k = 7$ and $n = 2$, we assign $f(u) = 4$, $f(v) = 1$, $f(u_1) = 0$, $f(u_2) = 5$, $f(v_1) = 2$ and $f(v_2) = 3$. From this labeling, we have $v_f(i) = \begin{cases} 0 & \text{if } i = 6 \\ 1 & \text{otherwise} \end{cases}$

and $e_f(i) = \begin{cases} 0 & \text{if } i = 4 \\ 1 & \text{otherwise} \end{cases}$ for $0 \leq i \leq 6$. Hence, the graph $S'(K_{1,2})$ is 7-product cordial.

Conversely, we assume that the graph $S'(K_{1,n})$ is k-product cordial if $\lfloor\frac{k}{3}\rfloor \leq n \leq \lfloor\frac{k}{2}\rfloor - 1$ where $k \equiv 1, 0\ (mod\ 3)$ for $k \geq 4$ except 7 and $\lceil\frac{k}{3}\rceil \leq n \leq \lfloor\frac{k}{2}\rfloor - 1$ for $k \equiv 2\ (mod\ 3)$ where $k \geq 8$. From this hypothesis, we have $v_f(i) = 1$ *or* 0, $e_f(i) = 1$ *or* 0 if $n = \lfloor\frac{k}{3}\rfloor$ *or* $\lceil\frac{k}{3}\rceil$ and $e_f(i) = 1$ *or* 2 if $\lfloor\frac{k}{3}\rfloor + 1 \leq n \leq$

$\lfloor \frac{k}{2} \rfloor - 1$ *or* $\lceil \frac{k}{3} \rceil + 1 \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$. Without loss of generality, we assign 1 and $k-1$ to the vertices u and v respectively. Also, we assign $\{0, 2, 3, ..., k-2\}$ to the remaining vertices u_i and v_i. Now, we have $e_f(1) = e_f(k-1) = 0$, which is a contradiction. Therefore, the graph $S'(K_{1,n})$ is not k-product cordial if $\lfloor \frac{k}{3} \rfloor \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ where $k \equiv 1, 0 \ (mod\ 3)$ for $k \geq 4$ except 7 and $\lceil \frac{k}{3} \rceil \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ for $k \equiv 2 \ (mod\ 3)$ where $k \geq 8$. Hence, the graph $S'(K_{1,n})$ is k-product cordial if and only if $n \geq 1$ and $k \geq 4$ except $\lfloor \frac{k}{3} \rfloor \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ if $k \equiv 1, 0 \ (mod\ 3)$ for $k = 4, 6$ *and* $k \geq 8$ and $\lceil \frac{k}{3} \rceil \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ if $k \equiv 2 \ (mod\ 3)$ for $k \geq 8$. □

Example 2.6. An example of 18-product cordial labeling of $S'(K_{1,5})$ shown in Figure 4.

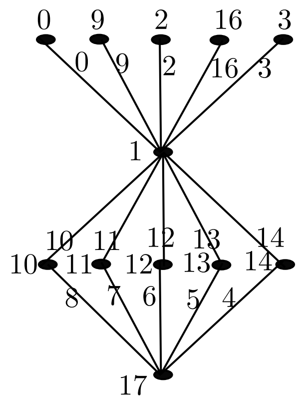

Figure 4: 18-Product cordial labeling of $S'(K_{1,5})$.

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