On Edge-Gracefulness of Wheel Graphs

Aaron DC. Angel, John Loureynz F. Gamurot, John Rafael M. Antalan
Department of Mathematics and Physics, College of Science, Central Luzon State University
Science City of Muñoz, Nueva Ecija, 3120, Philippines
angel.aaron@clsu2.edu.ph; gamurot.john@clsu2.edu.ph; jrantalan@clsu.edu.ph

ABSTRACT

The main focus of this paper is to present a complete proof that W_3 is the only edge-graceful wheel graph, a result that first appeared in the published work of S.Venkatesan and P.Sekar in 2017. In particular, we first discussed and exposed the results of S.Venkatesan and P.Sekar, regarding the edge-gracefulness of wheel graphs, and argue that their proof is incomplete. Then, we provide a complete proof of their result using the concepts of divisibility and Diophantine equations. We end the paper by giving some future works, related to edge-graceful labelling.

KEYWORDS: Graceful Labeling, Edge-graceful Labeling, Wheel Graph, Diophantine Equation

1 INTRODUCTION

The concept of **graph labeling** was first introduced in the mid-1960's. A graph labeling is an assignment of integers to the vertices or edges, or both, of a graph under certain conditions [1]. Within the last 60 years, over 200 types or variations of graph labeling have been studied and about 2500 articles have been published [2]. A **graceful labeling** or graceful numbering is a special graph labeling of a graph on m edges in which the vertices are labeled with a subset of distinct non-negative integers from 0 to m and each edge of the graph is uniquely labeled by the absolute difference between the labels of the vertices incident to it. If the resulting graph edge numbers run from 1 to m inclusive, it is a graceful labeling and the graph is said to be a graceful graph [3].

On the other hand, the concept of **edge-gracefulness** and edge-graceful graphs was defined and introduced by Sheng-Ping in 1985. Edge-graceful labeling is considered as a reversal of graceful labeling, because it labels the edges first, then the labels of the vertices would depend on the labels of the edges incident to them. That is, a graph G with p vertices and q edges, is said to be edge-graceful if the edges can be labeled from 1 through q, in such a way that the labels induced on the vertices by summing over incident edges modulo p are distinct [3].

In 2017, S.Venkatesan and P.Sekar wrote a paper regarding the edge-gracefulness of wheel graphs. They used the Lo's Theorem and Microsoft Excel to prove the main result of their paper, which claims that W_3 is the only edge-graceful wheel graph [3].

Motivated by their work, this research which aims to provide a complete proof of their result was created.

2 PRELIMINARIES

The following essential terms and results will be encountered and used in the succeeding sections of this article.

Definition 2.1. Graceful Labeling: A graceful labeling of a graph $G=(V,E)$ with p vertices and q edges is a 1-1 mapping f of the vertex set $V(G)$ into the set $\{0,1,2,...,q\}$. If we define, for any edge $e=(u,v) \in E(G)$, the value $\Omega(e)=|f(u)-f(v)|$, then Ω is a 1-1 mapping of the set $E(G)$ onto the set $\{1,2,...,q\}$. A graph is called graceful if it has a graceful labeling.

Figure 1 below, shows an example of a graph with graceful labelling. The notes beside it shows how it satisfies the definition of graceful labelling.

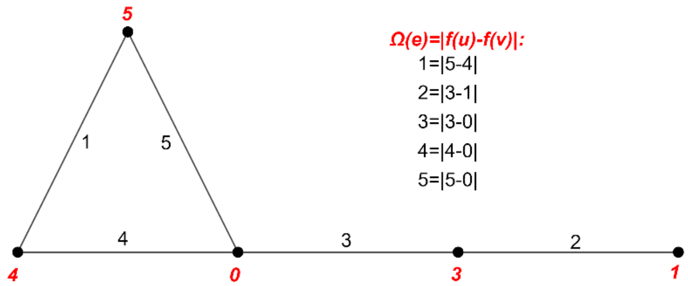

Figure 1. Example of Graceful Labeling

Definition 2.2. Edge-Graceful Labeling: A (p,q)-graph $G(V,E)$ is called edge-graceful if there exist a bijection $f: E \rightarrow \{1,2,....,q\}$ and the induced mapping $f^+ : V \rightarrow Z_p = \{0,1,....,p-1\}$ defined by

$$f^+ \equiv \sum_{uv \in E} f(uv)(mod\ p) \tag{1}$$

is a bijection and every $f(uv)$ are the labels of the edges incident to a corresponding vertex of G.

Figure 2 below, shows an example of a graph with edge-graceful labelling. The notes beside it shows how it satisfies the definition of edge-graceful labelling.

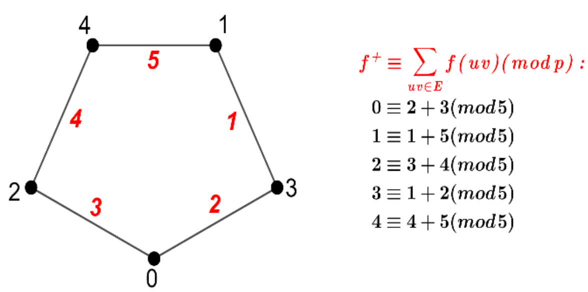

Figure 2. Example of Edge-Graceful Labeling

Definition 2.3. Wheel Graph: A wheel graph is the $(n+1,2n)$ graph which is formed from adding the middle vertex and edges, connecting the new vertex to every vertex of a cycle C_n . Wheels are denoted by Wn, where n refers to the number of vertices in the outer cycle.

For completeness, we recall some essential concepts in elementary Number Theory and Graph Theory.

Theorem 2.4. Handshaking lemma: In any graph, the sum of all the vertex degree is an even number.

Remark 2.5. We have the handshaking lemma since an edge always joins exactly two vertices in a graph.

Definition 2.6. Congruence: Let p be a positive integer. Also let $a,b \in \mathbb{Z}$. We can say that a is congruent to $b(modulo\ p)$ if $p|(a-b)$. That is if $a = b + kp$ for some $k \in \mathbb{Z}$.
Some important properties of congruence are given below.

Theorem 2.7. Let $a,b \in \mathbb{Z}$ and p be a positive integer. Then:

$$If\ a \equiv b(mod\ p), then\ b \equiv a(mod\ p) \tag{2}$$

Theorem 2.8. Suppose $a \equiv b(mod\ p)\ and\ c \equiv d(mod\ p), then\ a + c \equiv b + d(mod\ p)$

3 RESULTS

3.1 Exposition of S.Venkatesan and P.Sekar's Results

To begin this subsection, we first state and prove the Lo's Theorem, an initial condition to be satisfied for graph's edge-gracefulness.

Theorem 3.1. Lo's Theorem: If a graph G of p nodes and q edges is edge-graceful, then

$$\boldsymbol{p} \Big| \left(\boldsymbol{q^2 + q - \frac{p(p-1)}{2}} \right) \tag{3}$$

Proof:

Again, we note that a (p,q)-graph $G(V,E)$ is called edge-graceful if there exist a bijection $f : E \rightarrow \{1,2,....,q\}$ and the induced mapping $f^+ : V \rightarrow Z_p = \{0,1,....,p-1\}$ defined by (1),

$$f^+ \equiv \sum_{uv \in E} f(uv)(mod\ p)$$

is a bijection and every $f(uv)$ are the labels of the edges incident to a corresponding vertex of G.

In graph G, an edge connects exactly two vertices, by the **handshaking lemma**. We now consider all vertices of G, and by **Theorem 2.8**, we'll have,

$$\big(0 + 1 + 2 + \cdots . + (p-1)\big) \equiv 2(1 + 2 + \cdots . + q)(mod\ p) \tag{4}$$

By applying **Theorem 2.7** on (4), we'll have:

$$2(1 + 2 + \cdots . + q) \equiv \big(0 + 1 + 2 + \cdots . + (p-1)\big)(mod\ p) \tag{5}$$

$$2\sum_{x=1}^{q} x \equiv (0 + \sum_{x=1}^{p-1} x)(mod\ p) \tag{6}$$

Since $\sum_{x=1}^{N} \frac{N(N+1)}{2}$, we can simplify (6). That is,

$$2\left(\frac{q(q+1)}{2}\right) \equiv 0 + \left(\frac{(p-1)p}{2}\right)(mod\ p) \tag{7}$$

(8)

$$q^2 + q \equiv \left(\frac{(p-1)p}{2}\right)(mod\ p)$$

Which implies that,

$$\boldsymbol{p} | \left(\boldsymbol{q^2} + \boldsymbol{q} - \frac{\boldsymbol{p(p-1)}}{\boldsymbol{2}}\right)$$

■

Lo's Theorem is the initial condition for edge-gracefulness of a graph.

We now present the main results of S.Venkatesan and P.Sekar.

Theorem 3.2. (Main Result of S.Venkatesan and P.Sekar)
W_3 is the only edge-graceful wheel graph.

Proof:
First, a wheel graph W_n has *n+1* verities and *2n* edges. Using **Lo's Theorem**, if a graph with *p* vertices and *q* edges is edge-graceful, then

$$\boldsymbol{p} | \left(\boldsymbol{q^2} + \boldsymbol{q} - \frac{\boldsymbol{p(p-1)}}{\boldsymbol{2}}\right)$$

For W_n, we substitute *p=n+1* and *q=2n* in the theorem above. Thus,

$$(n+1) | \left((2n)^2 + 2n - \frac{(n+1)n}{2}\right) \tag{9}$$

By simplifying (9), we'll have:

$$(n+1) | \left(\frac{7n^2 + 3n}{2}\right) \tag{10}$$

Which implies that,

$$\left(\frac{7n^2 + 3n}{2(n+1)}\right) \in \mathbb{Z} \tag{11}$$

Hence, we must find the values of *n* that will make the above expression, an integer. Using Microsoft Excel-2021, we observe, which values of *n* from 1 to 200 (the original paper of S.Venkatesan and P.Sekar only used 1-100) will satisfy $\left(\frac{7n^2+3n}{2(n+1)}\right) \in \mathbb{Z}$ as shown in the figure below,

n	$(7n^2+3n)/(2(n+1))$	n	$(7n^2+3n)/(2(n+1))$	n	$(7n^2+3n)/(2(n+1))$	n	$(7n^2+3n)/(2(n+1))$	n	$(7n^2+3n)/(2(n+1))$
1	2.5	41	141.547619	81	281.52439	121	421.516393	161	561.512346
2	5.66666667	42	145.046512	82	285.024096	122	425.01626	162	565.01227
3	9	43	148.545455	83	288.52381	123	428.516129	163	568.512195
4	12.4	44	152.044444	84	292.023529	124	432.016	164	572.012121
5	15.8333333	45	155.543478	85	295.523256	125	435.515873	165	575.512048
6	19.2857143	46	159.042553	86	299.022989	126	439.015748	166	579.011976
7	22.75	47	162.541667	87	302.522727	127	442.515625	167	582.511905
8	26.2222222	48	166.040816	88	306.022472	128	446.015504	168	586.011834
9	29.7	49	169.54	89	309.522222	129	449.515385	169	589.511765

10	33.1818182	50	173.039216	90	313.021978	130	453.015267	170	593.011696
11	36.6666667	51	176.538462	91	316.521739	131	456.515152	171	596.511628
12	40.1538462	52	180.037736	92	320.021505	132	460.015038	172	600.011561
13	43.6428571	53	183.537037	93	323.521277	133	463.514925	173	603.511494
14	47.1333333	54	187.036364	94	327.021053	134	467.014815	174	607.011429
15	50.625	55	190.535714	95	330.520833	135	470.514706	175	610.511364
16	54.1176471	56	194.035088	96	334.020619	136	474.014599	176	614.011299
17	57.6111111	57	197.534483	97	337.520408	137	477.514493	177	617.511236
18	61.1052632	58	201.033898	98	341.020202	138	481.014388	178	621.011173
19	64.6	59	204.533333	99	344.52	139	484.514286	179	624.511111
20	68.0952381	60	208.032787	100	348.019802	140	488.014184	180	628.01105
21	71.5909091	61	211.532258	101	351.519608	141	491.514085	181	631.510989
22	75.0869565	62	215.031746	102	355.019417	142	495.013986	182	635.010929
23	78.5833333	63	218.53125	103	358.519231	143	498.513889	183	638.51087
24	82.08	64	222.030769	104	362.019048	144	502.013793	184	642.010811
25	85.5769231	65	225.530303	105	365.518868	145	505.513699	185	645.510753
26	89.0740741	66	229.029851	106	369.018692	146	509.013605	186	649.010695
27	92.5714286	67	232.529412	107	372.518519	147	512.513514	187	652.510638
28	96.0689655	68	236.028986	108	376.018349	148	516.013423	188	656.010582
29	99.5666667	69	239.528571	109	379.518182	149	519.513333	189	659.510526
30	103.064516	70	243.028169	110	383.018018	150	523.013245	190	663.010471
31	106.5625	71	246.527778	111	386.517857	151	526.513158	191	666.510417
32	110.060606	72	250.027397	112	390.017699	152	530.013072	192	670.010363
33	113.558824	73	253.527027	113	393.517544	153	533.512987	193	673.510309
34	117.057143	74	257.026667	114	397.017391	154	537.012903	194	677.010256
35	120.555556	75	260.526316	115	400.517241	155	540.512821	195	680.510204
36	124.054054	76	264.025974	116	404.017094	156	544.012739	196	684.010152
37	127.552632	77	267.525641	117	407.516949	157	547.512658	197	687.510101
38	131.051282	78	271.025316	118	411.016807	158	551.012579	198	691.01005
39	134.55	79	274.525	119	414.516667	159	554.5125	199	694.51
40	138.04878	80	278.024691	120	418.016529	160	558.012422	200	698.00995

Figure 3. Microsoft Excel-2021 Data

This method helped to show that from 1 to 200, only *n=3* will satisfy our conditions. However, this method is incomplete because it only considered some or limited values of *n*. Thus, we must use another method, where we need to consider all possible values of *n*. One way of doing so, is by using the concepts of divisibility and Diophantine equations.

3.2 An Alternative Proof, Using the Concepts of Divisibility and Diophantine Equations

In this subsection, we will completely prove that W_3 is the only edge-graceful wheel graph, using the Lo's theorem, and the concepts of divisibility and Diophantine equation.

Recall that using the **Lo's theorem**, if W_n with $p=n+1$ vertices and $q=2n$ edges is edge-graceful, then:

$$\left(\frac{7n^2+3n}{2(n+1)}\right) \in \mathbb{Z}$$

Hence,

$$7n^2+3n = 2(n+1)k, \quad For\ some\ k \in \mathbb{Z}$$
$$7n^2+3n = 2nk + 2k, \quad For\ some\ k \in \mathbb{Z}$$

Thus, we must solve for all the possible integer values of *n* and *k* in the Diophantine equation,

$$7n^2 + 3n - 2nk - 2k = 0, \quad where\ n, k \in \mathbb{Z} \tag{12}$$

Observe that this Diophantine equation is of the form,

$$an^2 + bnk + ck^2 + dn + ek + f = 0$$

Where *a=7*, *b=-2*, *c=0*, *d=3*, *e=-2*, and *f=0*.
The discriminant is $b^2 - 4ac = (-2)^2 - 4(7)(0) = \mathbf{4}$. Now, let D be the discriminant. we apply the **transformation of Legendre** [4], *Dn=X+*α, *Dk=Y+*β, and we will obtain:

$$\boldsymbol{\alpha} = 2cd - be = 2(0)(3) - (-2)(-2) = \mathbf{-4}\ \ and\ \ \boldsymbol{\beta} = 2ae - bd = 2(7)(-2) - (-2)(3) = \mathbf{-22}$$

So we will get the equation:

$$aX^2 + bXY + cY^2 = K \rightarrow \mathbf{7X^2 - 2XY = -64} \tag{13}$$

where $K = -D(ae^2 - bed + cd^2 + fD) = -64$.
The algorithm requires the constant coefficient to be positive, so we multiply both RHS and LHS of equation (13) by -1,

$$-7X^2 + 2XY = 64 \ \rightarrow\ (\mathbf{-7X + 2Y})(\mathbf{X}) = \mathbf{64} \tag{14}$$

We have to find all the factors of the RHS and solve for the resulting system of linear equations. We solve for ***n*** and ***k*** , where *D=4*, *Dn=X+*α, *Dk=Y+*β, α*=-4*, and β*=-22*. The figure below shows the summary of these results.

CASE	-7X+2Y	X	Y	$X+\alpha$	$Y+\beta$	n	k
1	1	64	-	-	-	no solution	no solution
2	-1	-64	-	-	-	no solution	no solution
3	2	32	113	28	91	no solution	no solution
4	-2	-32	-113	-36	-135	no solution	no solution
5	4	16	58	12	36	**3**	**9**
6	-4	-16	-58	-20	-80	**-5**	**-20**
7	8	8	32	4	10	no solution	no solution
8	-8	-8	-32	-12	-54	no solution	no solution
9	16	4	22	0	0	**0**	**0**
10	-16	-4	-22	-8	-44	**-2**	**-11**
11	32	2	23	-2	1	no solution	no solution
12	-32	-2	-23	-6	-45	no solution	no solution
13	64	1	-	-	-	no solution	no solution
14	-64	-1	-	-	-	no solution	no solution

Figure 4. Solutions of the Diophantine Equation

Therefore, we only have 4 possible solutions for the Diophantine equation $7n^2 + 3n - 2nk - 2k = 0$. These are: **(n,k)=(3,9), (-5,-20), (0,0), and (-2,-11)**. However, since we are only concern with possible values of n for wheel graph W_n, then we can limit the value of n, and that is, n must be a positive integer only. Hence, from our 4 solutions, only **(3,9)**, where ***n=3*** will satisfy this condition. Therefore, we have proven that **W_3 is the only wheel graph that satisfies the Lo's Theorem**.

For an illustrative example, the figure below is an example of edge-graceful labelling of a wheel graph W_3,

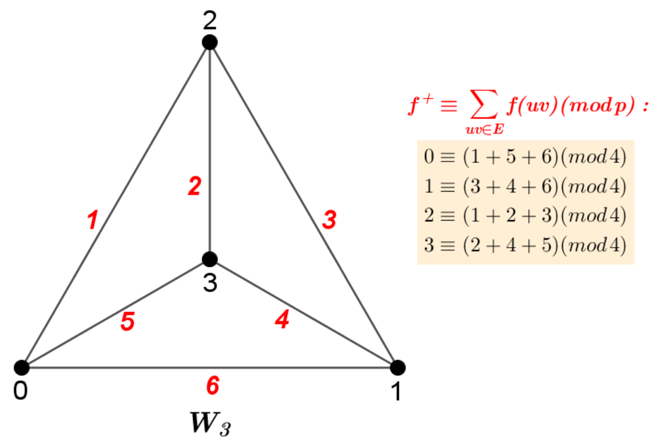

Figure 5. Edge-Graceful Labeling of W_3

Therefore, we can now conclude that W_3 is the only Edge-graceful wheel graph.

4 CONCLUSION

In this paper, we were able to present the results of S.Venkatesan and P.Sekar regarding W_3 as the only edge-graceful wheel graph. Using Lo's Theorem, they found out that if a wheel graph W_n is edge-graceful, then $\left(\frac{7n^2+3n}{2(n+1)}\right) \in \mathbb{Z}$. Using Microsoft Excel, they tested values of n from 1 to 100 (we extended it to 200), where they observed that only $n=3$ will satisfy the needed conditions. Thus, we were able to recognize that there is a limitation with their method.

Therefore, we provided an alternative and complete proof for their result using concepts of divisibility and Diophantine equations. In particular, we have shown that in order to prove the theorem, we must find the solutions to the Diophantine equation $7n^2 + 3n - 2nk - 2k = 0$. We solved that only **(n,k)=(3,9)** where **n=3** is the only solution that satisfies all our conditions. We were also able to provide an illustrative example of an edge-graceful labeling of W_3. Therefore, it proves that W_3 is the only edge-graceful wheel graph.

The readers are encouraged to explore other possible methods in proving the discussed results. Also, they are encouraged to use the concept of edge-gracefulness and explore other families/types of graphs.

5 ACKNOWLEDGEMENTS

We thank every individual and organizations who are responsible for the creation of this research paper. We thank the Central Luzon State University and the Department of Science and Technology-Science Education Institute, for their support.

REFERENCES

[1] J. Gallian, "A dynamic survey of graph labeling," *Electronic Journal of combinatorics*, vol. 1, 2018.

[2] S. Ramachandran and T. Gnanaseelan, "Prime labeling on some cycle related graphs," *Advances and Applications in Mathematical Sciences*, vol. 21, no. 12, pp. 6711-6719, 2022.

[3] S. Venkatesan and P. Sekar, "Not all wheel graphs were edge-graceful," *International Journal of Engineering, Science and Mathematics*, vol. 6, no. 5, 2017.

[4] K. Matthews, "Solving the diophantine Equation ax^2 + bxy + cy^2 + dx + ey + f = 0," *"unpublished"*, 2015.